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# The Sixth International Conference on Differential and Functional Differential Equations Moscow, Russia, August 14–21, 2011

# International Workshop "Spatio-temporal dynamical systems" Moscow, Russia, August 18–20, 2011

# ABSTRACTS

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Peoples' Friendship University of Russia
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**List of 45-minute Invited Lecturers:**
M. S. Agranovich, D. V. Anosov, S. Bianchini, V. M. Buchstaber, I. Capuzzo Dolcetta, A. A. Davydov, M. del Pino, N. Dencker, S. Yu. Dobrokhotov, Yu. A. Dubinskii, T. Fisher, H. Ishii, W. Jäger, V. V. Kozlov, P. Mironescu, S. P. Novikov, A. L. Piatnitski, S. I. Pohozaev, G. Sell, A. G. Sergeev, L. P. Shilnikov, A. E. Shishkov, A. A. Shkalikov, D. Shoikhet, Ya. G. Sinai, A. P. Soldatov, I. A. Taimanov, V. V. Vedenyapin, H.-O. Walther, J. Wu.

# Pulse Solutions of Some Hydrodynamical Problems: Existence and Stability

A. L. Afendikov
Keldysh Institute of Applied Mathematics, Moscow, Russia

We consider several hydrodynamic problems in unbounded domains where in the vicinity of the instability threshold the dynamics is governed by the generalized Cahn–Hilliard equation. For time-independent solutions of this equation, we recover Bogdanov–Takens bifurcation without parameter in the 3-dimensional reversible system with a line of equilibria. This line of equilibria is neither induced by symmetries, nor by first integrals. At isolated points, normal hyperbolicity of the line fails due to a transverse double eigenvalue zero. In case of bi-reversible problem, the complete set $B$ of all small bounded solutions consists of periodic profiles, homoclinic pulses, and a heteroclinic front-back pair (Asymptotic Analysis 60 (3, 4) (2008), 185–211). Later the small perturbation of the problem where only one symmetry is left was studied. Then $B$ consists entirely of trivial equilibria and multipulse heteroclinic pairs (Asymptotic Analysis, Volume 72, No. 1–2, 2011, pp. 31–76). Our aim is to discuss hydrodynamic problems, where the reversibility breaking perturbation can't be considered as small. We obtain the existence of a pair of heteroclinic solutions and partial results on their stability.

# Metrical Properties of Typical Homeomorphisms of Cantor Sets

O. N. Ageev
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Each homeomorphism of the Cantor set $K$ has at least one invariant probability measure, creating a measure-theoretical dynamical system. By fixing some measure $\mu$, we get a subset of $\mu$-preserving homeomorphisms, forming a Polish space in certain induced topology. We give the complete metrical and some topological descriptions of typical $\mu$-preserving homeomorphisms for natural $\mu$. We also treat the imbeddability of such homeomorphisms in free actions of different groups $G$ ($\mathbf{Z} < G$), by $\mu$-preserving transformations and homeomorphisms.

# Mixed Problems and Crack-Type Problems for Strongly Elliptic Second-Order Systems in Domains with Lipschitz Boundaries

M. S. Agranovich
Moscow Institute of Electronics and Mathematics, Russia

We consider two classes of problems for a strongly elliptic second-order system in a bounded $n$-dimensional domain with Lipschitz boundary, $n \geqslant 2$. For simplicity, we assume that the domain $\Omega = \Omega^+$ lies on the standard torus $\mathbb{T}^n$ and that the Dirichlet and Neumann problems in $\Omega^+$ and in the complementary domain $\Omega^-$ are uniquely solvable.

1. Mixed problems. In the simplest case, the boundary $\Gamma$ of $\Omega$ is divided into two parts $\Gamma_1$ and $\Gamma_2$ by a closed Lipschitz $(n-1)$-dimensional Lipschitz surface, with the Dirichlet and Neumann conditions on $\Gamma_1$ and on $\Gamma_2$ respectively. The problem is uniquely solvable in the simplest spaces $H^s$ (with the solution in $H^1(\Omega)$) and (the regularity result) in some more general Bessel potential spaces $H^s_p$ and Besov spaces $B^s_p$. Equations on $\Gamma$ are obtained equivalent to the problem. For this, we use analogs $N_1$ and $D_2$ of the Neumann-to-Dirichlet operator $N$ and the Dirichlet-to-Neumann operator $D$ on parts $\Gamma_1$ and $\Gamma_2$ of $\Gamma$.

The operators $N_1$ and $D_2$ are connected with Poincaré–Steklov-type spectral problems with spectral parameter on a part of $\Gamma$. In the selfadjoint case, the eigenfunctions form a basis in the corresponding spaces, and in the non-selfadjoint case they form a complete system. If $\Gamma$ is almost smooth (smooth outside a closed subset of zero measure), then the eigenvalues of self-adjoint problems have natural asymptotics.

2. Problems with boundary or transmission conditions on a non-closed surface $S$, which is a part of a closed Lipschitz surface $\Gamma$. In elasticity problems, $S$ is a crack, and in problems of acoustics and electrodynamics, it is a non-closed screen. The results are similar to those indicated above. The corresponding operators are restrictions $A_S$ to $S$ of the single layer potential-type operator $A$ and $H_S$ to $S$ of the hypersingular operator $H$ on $\Gamma$. For the corresponding spectral problems, the results are similar to those indicated above.

## On Behaviour of Trajectories Near Hyperbolic Sets

D. V. Anosov
Steklov Mathematical Institute, Moscow, Russia

Let $F$ and $F'$ be hyperbolic sets of diffeomorphisms $f$ and $f'$ respectively. Suppose that the restrictions $f|_F$ and $f'|_{F'}$ are topologically conjugated by a homeomorphism $h$. Then a restriction of $f$ to an invariant set comprised by all trajectories of $f$ that are close to $F$, and a restriction of $f'$ to a set of all trajectories of $f'$ that are close to $F'$, are conjugated by a homeomorphism $H$, which is an extenstion of $h$.

## Parabolic Obstacle-Type Problems with Contact Points

D. E. Apushkinskaya
Smirnov Scientific Research Institute in Mathematics and Mechanics, Saint-Petersburg, Russia

Many problems in physics, biology, finance, industry, and other areas can be described by partial differential equations that exhibit a priori unknown sets such as moving boundaries, interfaces, etc. The study of such sets, also known as free boundaries, often plays central role in the understanding of these problems.

In this talk we present a short survey on the special class of the parabolic free boundary problems, which is called the obstacle-type problems. The exact mathematical formulation is as follows:

Let function $u$ and an open set $\Omega \subset \mathbb{R}_+^{n+1}$ solve the problem

$$\begin{aligned} H[u] = \chi_\Omega \quad & \text{in} \quad Q_1^+, \\ u = |Du| = 0 \quad & \text{in} \quad Q_1^+ \setminus \Omega, \\ u = 0 \quad & \text{on} \quad \{x_1 = 0\} \cap Q_1, \end{aligned} \tag{1}$$

where $H = \Delta - \partial_t$ is the heat operator, $\chi_\Omega$ denotes the characteristic function of $\Omega$, $Q_1$ is the unit cylinder in $\mathbb{R}^{n+1}$, $Q_1^+ = Q_1 \cap \{x_1 > 0\}$, and the first equation in (1) is satisfied in the sense of distributions. We discuss different aspects of problem (1) near a fixed boundary, such as the optimal regularity of solutions, the study of blow-ups, and the regularity of the free boundary.

The talk is based on works in collaboration with Nina Uraltseva, Henrik Shahgholian, and Norayr Matevosyan.

# Dynamics of Stationary Structures in a Parabolic Problem with Reflected Spatial Argument

E. P. Belan

Vernadskii Taurida National University, Simferopol, Ukraine

We consider the boundary problem

$$\partial_t u(x,t) + u(x,t) = D\partial_{xx} u(x,t) + K(1 + \gamma \cos u(-x,t)), \ t > 0, \tag{1}$$

$$\partial_x u(-l,t) = \partial_x u(l,t) = 0. \tag{2}$$

Problem (1), (2) models the dynamics of phase modulation $u(x,t)$ of the light that passed through a thin layer of a nonlinear Kerr-type medium with a reflection transformation in the feedback loop in a one-dimensional approximation. Here $D > 0$, $K > 0$, and $0 < \gamma < 1$. We fix the smooth branches $w = w(K)$ of solutions of the equation $w = K(1 + \gamma \cos w)$. We fix some $K$ such that $\Lambda = \Lambda(K) = -K\gamma \sin w(K) < -1$. We denote $l_1 = \dfrac{\pi}{2}(\dfrac{-D}{1+l})^{1/2}$.

**Theorem 1.** *There is some $\delta > 0$ such that if we have $0 < l - l_1 < \delta$, then problem* (1), (2) *has the following two solutions:*

$$u_1^\pm = w \pm \left( \frac{D\pi(l - l_1)}{-c_1 l_1^3} \right)^{\frac{1}{2}} \sin \frac{\pi}{2l} x + O(l - l_1),$$

*where*

$$c_1 = -\frac{\Lambda}{4 \tan w} \left( (1 - \Lambda)^{-1} - \frac{1}{2}(3 + 5\Lambda)^{-1} \right) + \frac{\Lambda}{8}.$$

*Solutions $u_1^\pm$ are exponentially stable.*

In the case considered when $|\cos w| \ll 1$, it is shown that solutions $u_1^\pm$ preserve its stability with increasing $l$ when $\Lambda > -2$. If we have $\Lambda < -2$, then solutions

$u_1^{\pm}$ loses stability with increasing $l$. Other stationary structures bifurcate from $w$ as unstable solutions. However, they acquire stability with increasing $l$. If $\Lambda > -2$, then the stability acquired by structure is retained with the further increase in $l$. If we have $\Lambda < -2$, then each stationary structure retains stability in some variational interval $l$. After leaving the mentioned interval the instability index of the structure being considered unbounded increases with $l \to \infty$. We show that the number of stable stationary structures increases with $l \to \infty$.

### References


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## From Blow-up to Extinction for Solutions of Some Nonlinear Parabolic Equations


Y. Belaud
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Joint work with Carmen Cortázar (Pontificia Universidad Católica de Chile).

We consider a bounded regular domain $\Omega$ and the following equation:

$$\begin{cases} u_t - \Delta u = a(x)\, u^q - b(x) u^{q'} \text{ on } \Omega, \\ a, b \geqslant 0 \text{ a.e.},\ q > q' > 1, \end{cases} \tag{1}$$

for the Dirichlet boundary condition.

We try to characterize the behaviour of nonnegative solutions: do solutions have blow-up or extinction in a finite-time if functions $a$ and $b$ are regular or singular?

This work is in progress.

## A Direct Proof for the Selfadjointness of the Harmonic Oscillator


Ph. Berndt
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The harmonic oscillator is an important model system in quantum mechanics: using Taylor Expansion, stable equilibria of other systems can be locally expressed using its $q^2$-potential. From a mathematician's point of view, the oscillator is an operator

$$H\colon L^2 \to L^2,\ H\psi = \Delta\psi + q^2\psi,\ \operatorname{dom} H = \{\psi \in H^2 \colon q^2\psi \in L^2\}.$$

Operators of this kind are well understood. However, the theory covers a wider class of operators, making it hard for undergraduate students to understand it. Courses in functional analysis often only teach perturbation theory for Kato–Rellich-type and/or compact perturbations.

To encourage lecturers to also teach this specific example in their courses, I will present a simple extension to the ladder-method (typically used by physicists to determine eigenfunctions of $H$), showing selfadjointness and thus completing the operator's spectral analysis.

Based upon an idea by Caroline Lasser, TU München, and the bachelor theses of both Feliks Nüske, FU Berlin, and me, I will also point out how the oscillator can be used to descriptively demonstrate some deeper results.

## Regularity Estimates for Hamilton–Jacobi Equations and Hyperbolic Conservation Laws

S. Bianchini
International School for Advanced Studies, Trieste, Italy

Consider the Hamilton–Jacobi equation

$$u_t + H(\nabla u) = 0$$

with convex Hamiltonian. In spite of the fact that the Hamiltonian is only convex, and thus the characteristic vector field $d$ is in general not differentiable, we will show that the vector field $d$ has enough regularity to allow a change of variable formula.

Applications of this fact are a proof of the Sudakov theorem in optimal transportation theory and a solution of a conjecture of Cellina. In the case where $H$ is uniformly convex, we will show that the solution is not only semiconcave, but its first derivative is SBV.

We will also consider the hyperbolic system

$$u_t + f(u)_x = 0$$

and show that the direction of the characteristics are SBV.

## Decay Estimates and Singularities Hamilton–Jacobi Equation

M.-F. Bidaut-Veron
Université Francois-Rabelais de Tours, France

This work is made in collaboration with Anh Nguyen Dao.

Here we consider the nonnegative weak solutions of the parabolic problem

$$u_t - \Delta u + |\nabla u|^q = 0 \quad \text{in } \Omega \times (0,T), \tag{1}$$

where $q > 1$, $\Omega = \mathbb{R}^N$ or $\Omega$ is a bounded domain of $\mathbb{R}^N$, with irregular initial data $u_0$.

We show that a decay property and a regularizing effect occur for any weak solution $u$ in $\mathbb{R}^N$, when $u_0 \in L^r\left(\mathbb{R}^N\right)$, $r \geqslant 1$, or $u_0$ is a bounded Radon measure.

We give some extensions of these regularizing properties to quasilinear operators $\mathcal{A}$ under weak assumptions of coercivity.

We also study the problem in $\Omega$ with an eventual punctual singularity at $(x,t) = (0,0)$. We prove that for $q \geqslant \frac{N+2}{N+1}$, the singularity is removable. In particular there exists no very singular solution (V.S.S.) and no solution with a Dirac mass at $(0,0)$, showing that the well-known existence results for $q < \frac{N+2}{N+1}$ are optimal.

# Classical Solution of the Singularly Perturbed Free-Boundary Problem for the System of the Parabolic Equations

G. I. Bizhanova
Institute of Mathematics of Ministry of Education and Sciences, Kazakhstan

Multidimensional two-phase free-boundary problem for the system of the parabolic equations with two small parameters $\kappa > 0$ and $\varkappa > 0$ at the principal terms in the conditions on the free boundary is considered. The unique solvability and coercive estimates of the solutions of the problems with $\kappa > 0$, $\varkappa > 0$; $\kappa = 0$, $\varkappa > 0$ and $\kappa = 0$, $\varkappa = 0$ are obtained in the Hölder spaces locally in time.

# Elliptic Problems Coming from Supercollider Simulation

Ya. L. Bogomolov, E. S. Semenov, and A. D. Yunakovsky
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Electron (positron) accelerating structures are attractive to be fed with a wave flow converging onto the structure axis [1]. A proper structure proposed for future electron-positron colliders might represent a periodic set of coaxial radial-corrugated metallic discs exposed to a quasi-cylindrical wave flow [2]. Some symmetric model elliptic problems (boundary, inverse spectral, scattering) concerning synthesis of the optimal structure were considered earlier in [3].

The two main functional parts of the structure considered are (a) the cylindrically symmetrical paraxial working space, where the electromagnetic field interacts with injected particles, and (b) the non-symmetrical peripheral feeding region, where the incident electromagnetic wave is transformed into a convergent and properly phased symmetrical wave flow. Thus, the fundamental model problem arises: to transform the non-symmetrical wave flow coming from the angle feeding sector into the field structure as close to the convergent cylindrical wave as possible.

The model problem (in a whole space) is governed by the Helmholtz equation with an alternating wave number for various parts of the structure accompanying by radiation conditions in infinity together with continuous boundary conditions. To find an unknown solution, the method of discrete sources is used

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## Bianchi Cosmologies, the BKL Conjecture, and the Tumbling Universe

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In mathematical cosmology, one of the longstanding open questions is the structure of the initial singularity ("big bang") of the Einstein Equations. According to a conjecture of Belinskii, Khalatnikov, and Lifshitz (BKL) from the 1970s, the approach is vacuum-dominated, local, and oscillatory (labelled "Mixmaster" or the "Tumbling Universe"). This "BKL-picture" is supported by many heuristic and numerical calculations, but mathematical proofs exist only in very special cases. One important case are the spatially homogeneous Bianchi spacetimes (where the Einstein Equations reduce to ODEs), and even here the picture is far from complete, especially from a dynamical systems perspective.

One reason why the rigorous mathematical analysis of these ODEs is so difficult is the appearance of solutions with chaotic transient behavior described by dynamics of subshift type, which means that a dense set of periodic orbits as well as dense non-periodic orbits arise. This chaotic dynamics takes place near a circle of equilibria and its heteroclinic connections, which lead to the formation of (finite and infinite) heteroclinic chains. The main questions is how real solutions of the ODEs can be approximated by these (formal) heteroclinic chains when approaching the big bang. Another complication comes from the fact that naive linearization techniques fail due to the center direction that arises as a circle of equilibria plays a fundamental role in the dynamics.

We will discuss the current state of the art of rigorous convergence results in Bianchi cosmologies, where an emphasis will be put on those models that are important for more general PDE cosmological models. This can be seen as a first step towards making the BKL-picture more rigorous. According to the latter, the chaotic oscillations in Bianchi models are not only relevant for the (highly symmetric) spatially homogeneous cosmologies, but actually (generic) solutions to the full Einstein Equations are believed to have the same approximate behaviour — a Tumbling Universe at birth.

# Elliptic Functions, Differential Equations, and Dynamical Systems

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Denote by $\mathcal{E}_0$ the space of the universal bundle of elliptic curves with the space of parameters $g_2$, $g_3$ as base, and as fiber over the point $(g_2, g_3)$ the corresponding elliptic curve in the standard Weierstrass form with $t$ as coordinate. The field of Abelian functions of $t$ on $\mathcal{E}_0$ is determined by the Weierstrass function $\sigma(t; g_2, g_3)$, which is a section of the linear complex bundle over $\mathcal{E}_0$. The Weierstrass function $\wp = \wp(t; g_2, g_3) = -(\ln \sigma(t; g_2, g_3))''$ determines a birational equivalence $\mathcal{E}_0 \to \mathbb{C}^3$ : $(t, g_2, g_3) \to (\wp, \wp', \wp'')$, by which the differentiation along the fiber of the bundle $\mathcal{E}_0$ (the differentiation of functions along $t$) induces a classical algebraic dynamical system on $\mathbb{C}^3$. The algebra of differential operators along $t$, $g_2$, and $g_3$, which annihilate the $\sigma$-function, is extracted from classical works and leads to a solution of the well-known problem of differentiation of elliptic functions along parameters and, correspondingly, the problem of differentiation of a dynamical system solution along initial data. Using the generators of this algebra, we get dynamics in the space of parameters $g_2, g_3$, and on this basis the solution of the heat equation in terms of the $\sigma$-function. The dynamics are determined by a solution of the Shazy equation.

Let $\mathcal{E}_1$ be the space of the bundle with the space of parameters $g_2$, $g_3$ as base, and the fiber over the point $(g_2, g_3)$ the corresponding elliptic curve with coordinate $t$ and a marked point $\tau$. We obtain the bundle $\mathcal{E}_1 \to \mathcal{E}_0$ with the universal bundle of elliptic curves with parameter $\tau$ as base, and as fiber the elliptic curve with $t$ as parameter. The field of Abelian functions of $t$ and $\tau$ on $\mathcal{E}_1$ is determined by the function $\sigma(\tau; g_2, g_3)$ and the Baker-Akhiezer function $\Phi(t, \tau, g_2, g_3)$, which is a section of the linear complex bundle over $\mathcal{E}_1$. The function $\Phi(t, \tau, g_2, g_3)$ gives a solution of the Lame equation. It is a common eigenfunction of the Sturm-Liouville operator $\mathcal{L}_2$ with the potential $2\wp(t; g_2, g_3)$ and a third-order differential operator $\mathcal{L}_3$, which commutes with $\mathcal{L}_2$. The commutativity condition for the operators $\mathcal{L}_2$ and $\mathcal{L}_3$ is equivalent to the condition that the function $\wp$ is a solution of the stationary KdV equation.

We give differential equations on $\Phi(t, \tau, g_2, g_3)$, describing its dependence on parameters $g_2, g_3$. These equations completely determine the operators of differentiation of elliptic functions along the parameters. The function $P = -(\ln \Phi(t, \tau, g_2, g_3))'$ is elliptic along $t$ and $\tau$ and symmetric with respect to these variables. Using a differential equation on this function, we describe the algebraic surface $\mathcal{W}$ in $\mathbb{C}^5$ and a birational equivalence $\mathcal{E}_1 \to \mathcal{W}$, which is fiberwise with respect to a projection $\mathbb{C}^5 \to \mathbb{C}^3$. As a corollary, we obtain an algebraic dynamical system in $\mathbb{C}^5$ integrable in elliptic functions. We obtain three integrals of this system. We give differential equations that describe the dependence of a solution of the dynamical system on the initial data.

New results presented in the talk were obtained in recent joint works with E. Yu. Bunkova. The talk is addressed to a wide audience. Main definitions will be introduced during the talk.

# On the Vershik–Kerov Conjecture Concerning the Shannon–McMillan–Breiman Theorem for the Plancherel Family of Measures on the Space of Young Diagrams

A. I. Bufetov
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Vershik and Kerov conjectured in 1985 that dimensions of irreducible representations of finite symmetric groups, after appropriate normalization, converge to a constant with respect to the Plancherel family of measures on the space of Young diagrams. The statement of the Vershik–Kerov conjecture can be seen as an analogue of the Shannon–McMillan–Breiman theorem for the non-stationary Markov process of the growth of a Young diagram. The limit constant is then interpreted as the entropy of the Plancherel measure. The main result of the talk is the proof of the Vershik–Kerov conjecture. The argument is based on the methods of Borodin, Okounkov, and Olshanski.

The talk is based on the preprint arXiv:1001.4275

# On Expansions of Differential Operators in Banach Spaces

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It is well-known that the usual theory of partial differential operators expansions (Vishik, Hörmander, Berezansky, Dezin) or, which is equivalently, the general theory of boundary value problems has been building in the Hilbert space $L_2(\Omega)$. In this report a starting scheme of theory building for expansions in Banach spaces will be brought and initial results of the theory will be obtained.

In a bounded domain $\Omega \subset \mathbb{R}^n$ we consider expansions of operator (initially given in the space $C^\infty(\Omega)$) $\mathcal{L}^+ = \sum\limits_{|\alpha|\leqslant l} a_\alpha(x) D^\alpha$, $\quad D^\alpha = \frac{(-i\partial)^{|\alpha|}}{\partial x^\alpha}$ and its formal adjoint operator $\mathcal{L}^+ \cdot = \sum\limits_{|\alpha|\leqslant l} D^\alpha(a^*_\alpha(x)\ \cdot)$, where $a_\alpha(x)$ is an $N\times N^+$-matrix with entries $(a_\alpha)_{ij} \in C^\infty(\bar\Omega)$ and $a^*_\alpha(x)$ is the adjoint matrix.

For $p>1$ and $q=p/(p-1)$, we introduce graph norms $\|u\|_{L,p} = \|u\|_{L_p(\Omega)} + \|\mathcal{L}u\|_{L_p(\Omega)}$, $\|u\|_{L,q}$, $\|u\|_{L^+,p}$, and $\|u\|_{L^+,q}$. Then we build minimal operators $L_{p0}$, $L_{q0}$, $L^+_{p0}$, and $L^+_{q0}$ with its domains understood as the closures of $C^\infty_0(\Omega)$ in corresponding graph norms and maximal operators $L_p := (L^+_{q0})^*$, $L_q := (L^+_{p0})^*$, $L^+_p$, and $L^+_q$. Each operator $L_{pB} = L_p|_{D(L_{pB})}$ with property $D(L_{p0}) \subset D(L_{pB}) \subset D(L_p)$ is called an expansion (of $L_{p0}$), and the expansion $L_{pB} : D(L_{pB}) \to [L_p(\Omega)]^{N^+} =: B^+_p$ is called *solvable* if there exists its continuous two-sided inverse operator $L^{-1}_{pB} : B^+_p \to D(L_{pB})$, $L_{pB}L^{-1}_{pB} = \mathrm{id}_{B^+_p}$, $L^{-1}_{pB}L_{pB} = \mathrm{id}_{D(L_{pB})}$.

Here, as usually, one introduces the notion of the boundary-value problem in the form $L_p u = f$, $\Gamma u \in B$, where a subspace $B$ in the boundary space $C(L_p) =:$

$D(L_p)/D(L_{p0})$ ($\Gamma : D(L_p) \to C(L_p)$ is a factor-mapping) gives a homogenous boundary-value problem similar the Hörmander definition. Two Vishik conditions of the Hilbert case turn to four conditions in the Banach case: the operator $L_{p0}$ has a continuous left inverse (condition $(1_p)$) and the same refers to the operators $L_{q0}$ (condition $(1_q)$), $L^+_{p0}$ (condition $(1^+_p)$), and $L^+_{q0}$ (condition $(1^+_q)$). Then we prove the theorems:

**Theorem 1.** *The operator $L_{p0}$ has a solvable expansion if and only if conditions $(1_p)$ and $(1^+_q)$ are fulfilled.*

**Theorem 2.** *Under conditions $(1_p)$, $(1^+_p)$ we have a decomposition $D(L_p) = D(L_{p0}) \oplus \ker L_p \oplus W_p$, where $W_p$ is a subspace in $D(L_p)$ such that $L_p|_{W_p} : W_p \to \ker L^+_p$ is an isomorphism.*

**Theorem 3.** *Under conditions $(1_p)$, $(1^+_p)$ any solvable expansion $L_{pB}$ can be decomposed into a direct sum $L_{pB} = L_{p0} \oplus L^\partial_{pB}$, where $L^\partial_{pB} : B \to \ker L^{-1}_{p0}$ is an isomorphism.*

**Theorem 4.** *Under conditions $(1_p)$, $(1^+_p)$ any linear subspace $B \subset C(L_p)$ such that* (1) $\Gamma^{-1}_p B \cap \ker L_p = 0$ and (2) there exists operator $M_p : \ker L^{-1}_{p0} \to D(L_p)$ with the properties: (a) $L_p M_p = \mathrm{id}|_{\ker L^{-1}_{p\,0}}$ and (b) $\mathrm{Im} M_p \subset \Gamma^{-1}_p B$, generates a well-posed boundary-value problem (i.e. a solvable expansion $L_{pB}$ with domain $D(L_{pB}) = \Gamma^{-1} B$).

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# Glaeser-Type Interpolation Inequalities


I. Capuzzo Dolcetta

Sapienza University of Rome, Italy


I will present some recent results, in collaboration with A. Vitolo, concerning pointwise gradient estimates for nonnegative viscosity solutions of fully nonlinear second order elliptic equations. The results extend to a quite general nonlinear setting those of Yan Yan Li and Louis Nirenberg [*Progress in Nonlinear Differential Equations and Their Applications*, **66** (2005)] about the so-called Glaeser estimate for solutions of linear second-order elliptic equations.

# Homogeneous Dynamics for Theta Sums and Applications

F. Cellarosi
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I shall explain how limit theorems for theta sums can be obtained dynamically, by studying the equidistribution properties of horocycles under the action of the geodesic flow in a suitably defined hyperbolic manifold. I shall also present some applications to the study of auto-correlations in certain quantum mechanical systems.

# Solvability of a Generalized Buckley–Leverett Model

N. V. Chemetov
CMAF / University of Lisbon, Lisbon, Portugal

W. Neves
Institute of Mathematics, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil

We propose a new mathematical modelling of the Buckley–Leverett system, which describes the two-phase flows in porous media. We prove the solvability of the initial-boundary value problem for a deduced model

$$\partial_t u + \mathrm{div}\big(\mathbf{v}\ g(u)\big) = 0, \tag{1}$$
$$\tau\,\partial_t \mathbf{v} - \nu\Delta\mathbf{v} + h(u)\mathbf{v} = -\nabla p, \qquad \mathrm{div}(\mathbf{v}) = 0, \tag{2}$$

where $u = u(t,\mathbf{x})$ and $\mathbf{v} = \mathbf{v}(t,\mathbf{x})$ are the saturation and the total velocity of the two-phase flow. The parabolic/elliptic-type equations (2) are a generalized Darcy Law (Darcy–Brinkman law for $\tau \neq 0$ / Darcy–Forchheimer law for $\tau = 0$ ).

In order to show the solvability result, we consider an approximated parabolic-elliptic system. Since the approximated solutions do not have ANY type compactness property, the limit transition is justified by the kinetic method [1–3]. The main issue is to study a linear (kinetic) transport equation instead of the original nonlinear system.

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# Global Dynamics for Some Class of Fluid–Plate Interaction

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We study asymptotic dynamics of a coupled system consisting of linearized 3D Navier–Stokes equations in a bounded domain and the classical (nonlinear) elastic plate equation.

We consider two models for the plate oscillations:

(a) the model which account for transversal displacement on a flexible flat part of the boundary only, and

(b) the model for in-plane motions on a flexible flat part of the boundary.

In the latter case the main peculiarity is the assumption that the transversal displacements of the plate are negligible compared with in-plane displacements. This kind of models arises in the study of blood flows in large arteries.

For both cases our main results state the existence of a compact global attractor of finite dimension. Under some conditions this attractor is an exponentially attracting single point. We also show that the corresponding linearized system generates exponentially stable $C_0$-semigroup. We do not assume any kind of mechanical damping in the plate component. Thus, our results mean that dissipation of the energy in the fluid due to viscosity is sufficient to stabilize the system.

In the case (a) the result has been obtained in collaboration with I. L. Ryzhkova.

# Semilinear Hyperbolic Functional Differential Problem on a Cylindrical Domain

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We consider the initial-boundary value problem for a semilinear partial functional differential equation of the first order on a cylindrical domain in $n + 1$ dimensions. Projection of the domain onto the $n$-dimensional hyperplane is a connected set with boundary satisfying certain type of cone condition. Using the method of characteristics and the Banach contraction theorem, we prove the global existence, uniqueness, and continuous dependence on data of Caratheodory solutions of the problem. This approach covers equations with deviating variables as well as integrodifferential equations.

# On a Spectral Problem for an Ordinary Differential Operator with Integral Conditions

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A second-order ordinary differential equation with a spectral parameter and integral conditions is considered. An a priori estimate of the solution for sufficiently large values of the parameter is obtained and spectral properties of the corresponding operator are studied.

# Optimization of Steady State of Forest Management Model

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Choosing a measurable control $u$, $0 \leqslant u_1(l) \leqslant u \leqslant u_2(l)$, we optimize the steady state solution of the forest model [1]

$$\frac{\partial x(t,l)}{\partial t} + \frac{\partial g(l,x(t,.))x(t,l)}{\partial l} = -m(l,x(t,.))x(t,l), \tag{1}$$

where $x := x(t,l)$ is the density of biomass of size $l$ at the moment $t$, $m(l,x(t,.)) = \mu(l,x(t,.))+u(l,x(t,.))$, $g$ and $\mu$ are rates of the biomass death and growth respectively, $u$ is the biomass portion harvested at size $l$, and $L$ is the size of the clear-cutting. We assume that $x(t,0) = \int_{l_0}^{L} r(l)x^{\beta}(t,l)dl + p(t)$, where $l_0 > 0$ is the minimum reproductivity size, $\beta$ is an elasticity coefficient, $r, r = r(l)$, is the rate of biomass reproductivity at size $l$, and the rate is zero and greater than zero if $0 \leqslant l < l_0$ and $l_0 < l \leqslant L$ respectively.

We show that under reasonable assumptions on the model parameters every nonzero solution (1) converges to a steady-state solution, for which the biomass density does not depend on time, i.e. $x = x(l)$, if $p(t) \equiv p_0 > 0$, $g = g(l)$, $\mu = m(l)$ and $u = u(l)$. That reduces the optimization problem to the selection of control $u$, maximizing profit $\int_0^L u(l)c(l)x(l)dl - p_0c_0$ on this solution. Here $c, c = c(l)$, and $c_0$ are the aggregated prices of harvested biomass at size $l$ and biomass planting at size 0, respectively. Under reasonable assumption on the model parameters we prove that the optimal solution exists and is unique. Also, we find a necessary optimality condition being similar to the condition in the paper [2].

The work was completed by partial financial support of RFBR (№ 10-01-91004-ASF-a) и ADTP HSSPD (№ 2.1.1/12115) grants.

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# On the Existence of Wheeler–Feynman Electrodynamics

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The equations of Wheeler–Feynman electrodynamics are given by a set of functional-differential equations involving state-dependent retarded and advanced arguments of unbounded delay. In the case of two particles of equal charge and when the motion is restricted to a straight line G. Bauer proved the existence of solutions which are characterized by their asymptotic properties. In a joint work with G. Bauer and D. Dürr we studied the existence of solutions for given Newtonian Cauchy data for $N$ charges without geometrical restrictions in three dimensions. Neglecting collision singularities, we show existence of charge trajectories which fulfill the Wheeler–Feynman equations on arbitrary large time-intervals. As a byproduct we give a simple proof of the existence of solutions to the Synge equations on the time half-line.

# New Entire Solutions to Semilinear Elliptic Equations

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We will survey some recent results on the construction of entire solutions of semilinear elliptic equations. We will mostly focus on the construction of families of solutions to the Allen–Cahn equation of phase transitions, whose level sets suitable scaled concentrate around a given minimal surface. To do so, we introduce an infinite-dimensional form of Lyapunov–Schmidt reduction suitable for this and several related questions.

# Spectral Problems of Waveguide Theory and Keldysh Operator Pencil

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Maxwell equations

$$rotE = ik\mu(x,y)H, \qquad rotH = -ik\varepsilon(x,y)E, \tag{1}$$

$$div\varepsilon(x,y)E = 0, \qquad div\mu(x,y)H = 0 \tag{2}$$

are considered in a cylindrical domain $Q = \{(x,y) \in \Omega, z \in (-\infty, \infty)\}$. The investigation of such solution as $E = E(x,y)e^{i\gamma z}, H = H(x,y)e^{i\gamma z}$ leads to a spectral problem where $\gamma$ is an eigenvalue and $E(x,y), H(x,y)$ are eigenvectors. Different formulations of the problem were considered when the system of equations (1) was applied for reduce Maxwell equations to a boundary problem for second-order equations [1–3]. The proof of completeness of the eigenvectors may be done only for some special cases when we use such formulations. In [4, 5], we suggest to use another approach for studying this problem, connected with system (2) and equations of system (1) with $\frac{\partial}{\partial z}$ derivatives. This permits us to derive the problem of eigenvectors completeness to the consideration of a boundary problem for equations

$$-grad_{\perp}\mu^{-1}div_{\perp}B_{\perp} - k^2\varepsilon B_{\perp} - ik\varepsilon rot_{\perp}E_z = -\gamma^2\mu^{-1}B_{\perp} \tag{3}$$

$$-ikrot_{\perp}\varepsilon^{-1}B_{\perp} - div_{\perp}\varepsilon^{-1}grad_{\perp}E_z = -\gamma^2\varepsilon E_z, \tag{4}$$

where $D_{\perp} = \varepsilon E_{\perp}$, $B_{\perp} = \mu H_{\perp}$. The investigation of the spectral problem for equations (3)-(4) may be reduced in the appropriate functional space to the application of Keldysh theory of operator pencils. Now we prove the completeness of eigenvectors of another formulations which founded on the mentioned theorem.

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# The Solvability of Differential Equations

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It was a great surprise when Hans Lewy in 1957 presented a non-vanishing complex vector field that is not locally solvable. Actually, the vector field is the tangential Cauchy–Riemann operator on the boundary of a strictly pseudoconvex domain. Hörmander proved in 1960 that almost all linear partial differential equations are not locally solvable, because the necessary bracket condition is non-generic. This also has consequences for the spectral instability of non-selfadjoint semiclassical operators and the solvability of the Cauchy problem for non-linear analytic vector fields.

Nirenberg and Treves formulated their famous conjecture in 1970: that condition $(\Psi)$ is necessary and sufficient for the local solvability of differential equations of principal type. Principal type essentially means simple characteristics, and condition $(\Psi)$ only involves the sign changes of the imaginary part of the highest order terms along the bicharacteristics of the real part.

The Nirenberg–Treves conjecture was finally proved in 2003. We shall present the background, the main results, and some generalizations to non-principal type equations and systems of differential equations.

# On Continuous Invertibility and Fredholm Property of Differential Operators with Multivalued Impulse Effects

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Let $X$ be a Banach space over the field $\mathbb{R}$ or $\mathbb{C}$. Let $\operatorname{End} X$ be the Banach algebra of all endomorphisms of the space $X$. Let us consider a segment $[t_0, t_2]$ and a point $t_1$ from this segment. We denote by the symbol $\widetilde{C} = \widetilde{C}([t_0, t_2], X)$ the Banach space of all functions that are continuous on the sets$[t_0, t_1]$ and $(t_1, t_2]$ and have finite left-hand limit $x^+(t_1)$ at the point $t_1$. We endow this space with the norm

$$\|x\| = \sup_{t \in [t_0, t_2]} ||x(t)||.$$

The symbol $\Delta$ denotes the set $[t_0, t_2] \times [t_0, t_2]$. A map $\mathcal{U} : \Delta \to \operatorname{End} X$ is called a (strongly continuous) family of evolution operators on $[t_0, t_2]$ if the following conditions are fulfilled:

(1) $\mathcal{U}(t, t) = I$ is the identity operator for all $t \in [t_0, t_2]$;

(2) $\mathcal{U}(t, s)\mathcal{U}(s, \tau) = \mathcal{U}(t, \tau)$ for all $t, s, \tau$ in $[t_0, t_2]$;

(3) the map $(t, s) \longmapsto \mathcal{U}(t, s)x : [t_0, t_2] \times [t_0, t_2] \to X$ is continuous for all $x \in X$.

Let $\mathcal{A}$ and $\Gamma$ be closed linear relations (multivalued operators) on the space $X$. We define the operator $\mathcal{L} : D(\mathcal{L}) \subset \widetilde{C} \to \widetilde{C}$ as follows. A function $x \in \widetilde{C}$ such that $(x(t_0), x(t_2)) \in \Gamma$ and $(x(t_1), x^+(t_1)) \in \mathcal{A}$ belongs to $D(\mathcal{L})$ if there exists a function $f \in \widetilde{C}$ such that

$$x(t) = \mathcal{U}(t, s)x(s) + \int_s^t \mathcal{U}(t, \tau)f(\tau)d\tau, \quad t_{i-1} < s \leqslant t \leqslant t_i, \qquad i = 1, 2.$$

In this case we set $\mathcal{L}x = f$.

We denote by the symbol $\mathcal{D}$ the relation $\Gamma - \mathcal{U}(t_2, t_1)\mathcal{A}\mathcal{U}(t_1, t_0)$, by the symbol $\mathcal{X}_1$ the subspace $\Gamma 0 \cap \mathcal{U}(t_2, t_1)\mathcal{A}0$, and by the symbol $\mathcal{X}_2$ the subspace $\mathcal{U}(t_1, t_0)D(\Gamma) + D(\mathcal{A})$.

**Theorem 1.** *The operator $\mathcal{L}$ is continuously invertible if and only if the relation $\mathcal{D}$ is continuously invertible, $\mathcal{X}_1$ consists of zero only, and $\mathcal{X}_2$ coincides with the whole $X$.*

**Theorem 2.** *The operator $\mathcal{L}$ possesses the Fredholm property if and only if the relation $\mathcal{D}$ possesses the Fredholm property, $\mathcal{X}_1$ has a finite dimension, and $\mathcal{X}_2$ has a finite codimension.*

# Quantum Double Well in Magnetic Field: Tunnelling, Libration, Normal Forms

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We consider the spectral problem for 2-D magnetic Shrödinger operator with the potential having a form of symmetric double well and with constant magnetic field. This problem is very well studied when the magnetic field is absent. In particular, in semiclassical limit the distance between two lowest eigenvalues is exponentially small with respect to a corresponding small parameter $h$. The derivation of explicit formulas for the splitting of eigenvalues (V. Maslov, A. Poljakov, E. M. Harrel, B. Helfer, J. Sjostrand, B. Simon, etc.) is based on a passage from standard fast oscillating WKB-functions $A(x)e^{\frac{iS(x)}{h}}$ to fast decaying functions $A(x)e^{\frac{-S(x)}{h}}$. This passage changes the corresponding real-valued standard Hamiltonian $H = p^2/2+V(x)$ to (again!) the real-valued "tunnelling" Hamiltonian $H = -p^2/2 + V(x)$, which allows one to use the theory of the classical Hamiltonian systems for description of tunnel effects. This idea does not work for the situation including the magnetic field: the "tunnelling" Hamiltonian becomes a complex-valued function and one cannot use the theory of the classical Hamiltonian systems. Our observation is that one can reduce the quantum double-well problem for the magnetic Shrödinger operator to the standard quantum double-well problem using the partial Fourier transform and mixed momentum-position coordinates. We show also that the splitting formula takes natural and simple form if it is based on so-called libration and normal forms coming from classical mechanics. We apply these results for description of tunnelling of wavepackets in quantum nanowires.

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# Choice Problem of Weight Functions in Hardy-Type Inequalities and Applications to PDEs in Full Euclidean Space

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We consider the following questions:

1. Constructive description of all possible weighting functions in Hardy-type inequalities;
2. Multidimensional weighted inequalities of the Friedrichs and Poincaré type in full Euclidean space;
3. Elliptic equations in the Sobolev scale of functions in the full Euclidean space;
4. Decomposition of the Sobolev spaces and gradient-divergence spaces in the sum of solenoidal and potential subspaces;
5. Divergence and rotor variants of the Stokes systems in the full Euclidean space ("explicit" solutions);
6. Stationary Fokker–Plank–Kolmogorov equations (continuum of nontrivial solutions).

# On Initial-Boundary Value Problems for Odd-Order Quasilinear Evolution Equations

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Initial-boundary value problems in the domains $\Pi_T^+ = (0,T)\times\mathbb{R}^n_+$, $\mathbb{R}^n_+ = \{x : x_n > 0\}$, $\Pi_T^- = (0,T)\times\mathbb{R}^n_-$, $\mathbb{R}^n_- = \{x : x_n < 0\}$, and $Q_T = (0,T)\times\Sigma$, $\Sigma = \{x : 0 < x_n < 1\}$, where $x = (x_1,\dots,x_n)$ and $T > 0$ are arbitrary, are considered for an equation

$$u_t - \sum_{k=0}^{2l+1} P_k(\partial_x)u + \operatorname{div}_x g(u) = f(t,x), \qquad l \in \mathbb{N},$$

where $u = u(t,x)$, $g = (g_1,\dots,g_n)$, and $P_k$ are linear homogeneous differential operators of orders $k$. Initial and boundary conditions

$$\begin{gathered} u\big|_{t=0} = u_0(x), \\ \partial_{x_n}^m u\big|_{\partial^l\Omega} = \mu_l(t,x'), \qquad m = 0,\dots,l-1, \\ \partial_{x_n}^m u\big|_{\partial^r\Omega} = \nu_l(t,x'), \qquad m = 0,\dots,l, \end{gathered}$$

for $x \in \Omega$, $(t, x') \in B_T = (0,T) \times \mathbb{R}^{n-1}$, $x' = (x_1, \ldots, x_{n-1})$, where $\Omega$ is either $\mathbb{R}^n_+$, $\mathbb{R}^n_-$ or $\Sigma$ respectively and $\partial^l \Omega$, $\partial^r \Omega$ are respectively the left-hand and the right-hand parts of the boundary (if exist), are set. The operators $P_k$ are subjected to the following two assumptions:

$$(1) \quad (-1)^l \frac{\partial}{\partial \xi_n} P_{2l+1}(\xi) > 0 \qquad \forall \xi \neq 0,$$

$$(2) \quad \text{either } (-1)^j P_{2j}(\xi) \leqslant 0 \qquad \forall \xi \in \mathbb{R}^n, \quad j = 1, \ldots, l,$$

or there exists natural $m$, $2 \leqslant m \leqslant l$, such that

$$(-1)^m P_{2m}(\xi) < 0 \quad \forall \xi \neq 0, \qquad (-1)^j P_{2j}(\xi) \leqslant 0 \quad \forall \xi \in \mathbb{R}^n, \quad j = m+1, \ldots, l,$$

where $P_k(\xi)$ are the symbols of the operators $P_k$. The functions $g_j(u)$ are assumed to be in the space $C^1(\mathbb{R})$ and satisfy the following growth restrictions: for all $u \in \mathbb{R}$

$$|g_j'(u)| \leqslant c(|u|^{b_j} + 1), \quad 0 \leqslant b_j \leqslant 1, \quad b_n < 4l/n.$$

Results on existence and uniqueness of global weak solutions are established. For example, the following theorem is proved for the third problem.

**Theorem 1.** *Assume that $u_0 \in L_2(\Sigma)$, $f \in L_1(0,T; L_2(\Sigma))$, $\mu_m, \nu_m \in H^{(s+l-m)/(2l+1), s+l-m}_{t,x'}(B_T)$, where $s = 0$ if $n < (2l-1)$ and $s > (n+1)/2 - l$ if $n \geqslant (2l-1)$, $m = 0, \ldots, l-1$, $\nu_l \in L_2(B_T)$. Then there exists a weak solution $u(t,x)$ to the considered problem in $Q_T$ such that*

$$u \in C_w([0,T]; L_2(\Sigma)) \cap L_2(0,T; H^l(\Sigma)).$$

*If, in addition, $b_j \leqslant (4l-2)/n$ for all $j$, then the constructed solution is unique in such a class.*

# Entropic Stability

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Andronov and Pontryagin suggested that the study of dynamical systems should focus on stable systems. It turns out that topologically C1-stable dynamics (also called structurally stable systems) can be analyzed; indeed, they are exactly the uniformly hyperbolic systems that satisfy an additional assumption. However, the structurally stable diffeomorphisms are not dense and therefore the study of such systems is insufficient. A natural approach is to consider weaker forms of stability. In this talk we will introduce an entropy-based notion of stability. Furthermore, we analyze a class of deformations of Anosov diffeomorphisms: these deformations break the topological conjugacy class but leave the high entropy dynamics unchanged. More precisely, there is a partial conjugacy between the deformation and the original Anosov system that identifies all invariant probability measures with entropy close to the maximum.

# Synchronizability of Networks with Strongly Delayed Links: a Universal Classification

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Stability of synchronization in delay-coupled networks of identical units generally depends in a complicated way on the coupling topology. We show that for large coupling delays synchronizability relates in a simple way to the spectral properties of the network topology. The master stability function used to determine the stability of synchronous solutions has a universal structure in the limit of large delay: It is rotationally symmetric around the origin and increases monotonically with the radius in the complex plane. This allows a universal classification of networks with respect to their synchronization properties and solves the problem of complete synchronization in networks with strongly delayed coupling.

# Decentralized Output Feedback Synchronization of Dynamical Networks

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Controlled synchronization of networks has a broad area of important applications: control of power networks, cooperative control of mobile robots, control of lattices, control of biochemical, ecological networks, etc. However, most existing papers deal with the networks of dynamical systems with full state measurements and full control (vectors of agent input, output, and state have equal dimensions). In the talk a survey of authors' results on synchronization of nonlinear dynamical networks with incomplete measurements and incomplete control in presence of possible delays and disturbances is given.

Let the network $S$ consist of $N$ interconnected subsystems (agents)

$$\dot{x}_i = Ax_i + Bu_i + \varphi(x_i) + \sum_{j=1}^{N} \{\alpha_{ij}\varphi_{ij}(x_i - x_j) + \beta_{ij}\psi_{ij}(x_j(t-\tau) - x_i(t-\tau))\} + f_i(t), \quad (1)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^l$. Functions $\varphi_{ij}(\cdot), \psi_{ij}(\cdot) i = 1, \ldots, N, j = 1, \ldots, N$ describe either physical or communication links among the subsystems and $f_i(t)$ is some bounded disturbance. Let the control goal be

$$\overline{\lim_{t\to\infty}} |x_i(t) - \overline{x}(t)| \leqslant \Delta_i, \quad (2)$$

where $\overline{x}(t), \overline{u}(t)$ are the state and the input of the leader agent

$$\dot{\overline{x}} = A\overline{x} + B\overline{u}(t) + \varphi(\overline{x}), \overline{y} = C^{\mathrm{T}}\overline{x}. \quad (3)$$

The goal is to find decentralized control $u_i = U_i(y_i, t)$, providing (2). The proposed adaptive control algorithms typically have the form

$$\tilde{u}_i = \theta_i(t)^{\mathrm{T}} \tilde{y}_i,\ \theta_i(t) \in \mathbb{R}^l,\ i = 1, \ldots, N, \tag{4}$$

$$\dot{\theta}_i(t) = \begin{cases} -g^{\mathrm{T}} \tilde{y}_i(t) \Gamma_i \tilde{y}_i(t),\ Q_i(x_i(t), t) > \Delta_i \\ 0\ ,\ Q_i(x_i(t), t) \leqslant \Delta_i. \end{cases} \tag{5}$$

where $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$ are $l \times l-$matrices and $\theta_i(t)$ are adjustable parameters. The conditions ensuring control goal are established. The work is supported by RFBR (11-08-01218) and FGP "Cadres" (contracts 16.740.11.0042, 14.740.11.0942).

# Parabolic Equations of Normal Type: Structure of Phase Flow and Nonlocal Stabilization

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Energy estimates play an important role in the investigation of 3D Navier–Stokes systems and other equations of continuous media. The absent of such bounds in the phase space $H^1$ is a very serious obstacle to prove the existence of nonlocal smooth solutions.

A semilinear parabolic equation is called equation of normal type if its nonlinear term $B$ satisfies the following condition: the vector $B(v)$ is collinear to the vector $v\ \forall v$. Since energy estimates are derived from the condition $B(v) \perp v$, equations of normal type does not satisfy the energy bound "in the most degree". That is why their investigation should help to clarify certain questions related to energy estimates.

For the parabolic equation of normal type with periodic boundary conditions, the structure of its phase flow will be described. Its phase space can be divided on: (a) the stability set (solutions with initial conditions from this set tends to zero as the time $t \to \infty$ with a certain prescribed rate, (b) the set of explosions where the solution blows up during a finite time, and (c) the intermediate set dividing previous ones, where the solution either tends to zero as $t \to \infty$ slower than solutions from the set (a) or it increases unboundedly as $t \to \infty$.

For arbitrary initial condition, we construct starting control supported in a given fixed subset such that the solution of the obtained boundary-value problem tends to zero as $t \to \infty$. The structure of the control becomes universal: only its norm and "sign" depend on the initial condition. As known, this starting control can be used for construction of the control on the boundary (in the case of the mixed Dirichlet boundary-value problem) that stabilizes the solution of the mentioned boundary-value problem.

# Monotonicity and Nonexistence for Quasilinear Dirichlet Problems in a Half-Space

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We consider the problem

$$\begin{cases} -\Delta_p u = u^q & (x \in \mathbb{R}^N_+), \\ u > 0 & (x \in \mathbb{R}^N_+), \\ u = 0 & (x \in \partial\mathbb{R}^N_+), \end{cases} \tag{1}$$

where $\Delta_p u := \operatorname{div}(|Du|^{p-2}Du)$, $p > 1$, $q > \max\{1, p-1\}$.

We establish new sufficient conditions for monotonicity of solutions to problem (1) with respect to the normal variable. This allows us to obtain new nonexistence results for this problem.

The proof is based on a new version of the comparison principle for quasilinear elliptic operators in unbounded domains and on the technique of moving planes.

# On Regge–Gel'fand Problem of Construction of the Pfaff System of Fuchsian Type with a Given Singular Divisor

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At the beginning of the sixties of the XX century I. M. Gelfand stated the problem of construction of the system of partial differential equations of generalized hypergeometric type for Feynman integrals of quantum electrodynamics. T. Regge presented it on International Conference "Battelle rencontres". The definition of equations of the generalized hypergeometric type was not given. One of treatment of this statement was the consideration of the problem as the Riemann–Hilbert problem of construction of differential equations using the ramification of a given object and its monodromy group. At this moment (beginning from the twenties) there were known several hypergeometric functions (Appell and Kampe de Feriet) $F_1, F_2, F_3, F_4$ of two complex variables. There was natural to try to write hypergeometric-type equations for them, using the similarity with one-dimensional case. This was done. Next step, using the fact that the Feynman integrals have singular points on the Landau varieties (later the ramification type of these integrals was investigated) and using a theorem of algebraic geometry on reduction of order of pole, the corresponding systems of partial differential equations was written. As in the case of Appell functions of, those equations were of the Fuchsian type. So, the form of desired equations became obvious. From the other hand, physicists prepared for mathematicians very reach class of partial differential equations of Fuchsian type: Knizhnik–Zamolodchikov equations associated with the root system and the ramification along the reflection hyperplanes of these systems. In the structure of the coefficients of such systems the role play the Casimir elements of corresponding Lie algebra. These invariants of the second order and also of higher orders play the principal role in construction of the Fuchsian type equations in two or more parametric case, where as parameters are considered constants number of which is equal to the number of orbits of the root system. It is of great interest the Fuchsian reduction of the nonlinear partial differential equations.

In the talk the results obtained in this direction and open problems will be considered.

## Dynamical Properties of the Trace Map and Spectrum of the Weakly Coupled Fibonacci Hamiltonian

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It is always exciting to obtain a new connection between two different areas of mathematics. It turns out that there is a beautiful relation between the spectral properties of the discrete Schrödinger operator with Fibonacci potential, the so-called Fibonacci Hamiltonian, and the modern theory of dynamical systems (namely, uniformly hyperbolic and normally hyperbolic dynamics).

The Fibonacci Hamiltonian is a central model in the study of electronic properties of one-dimensional quasicrystals. It is given by the following bounded selfadjoint operator in $\ell^2(\mathbb{Z})$:

$$[H_{V,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + V\chi_{[1-\alpha,1)}(n\alpha + \omega \mod 1)\psi(n),$$

where $V > 0$, $\alpha = \frac{\sqrt{5}-1}{2}$, and $\omega \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$.

We consider the spectrum of the Fibonacci Hamiltonian (it is known that this set is a Cantor set of zero Lebesgue measure) for small values of the coupling constant, and study the limit, as the value of the coupling constant approaches zero, of its thickness and its Hausdorff dimension. We prove that the thickness tends to infinity and, consequently, the Hausdorff dimension of the spectrum tends to one. We also show that at small coupling, all gaps allowed by the gap labelling theorem are open and the length of every gap tends to zero linearly. Moreover, for sufficiently small coupling, the sum of the spectrum with itself is an interval. This last result provides a rigorous explanation of a phenomenon for the Fibonacci square lattice discovered numerically by Even-Dar Mandel and Lifshitz [6,7]. Finally, we show that the density of states is exact-dimensional, and its dimension also tends to one as coupling constant tends to zero. The proofs of these results [4,5] are based on hyperbolicity of the trace map associated with Fibonacci Hamiltonian [1–3].

This is a joint work with David Damanik.

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# Reaction-Diffusion Equations with Hysteretic Free Boundary

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We consider reaction-diffusion equations involving a hysteretic discontinuity which is defined at each spatial point. These problems describe chemical reactions and biological processes in which diffusive and nondiffusive substances interact according to hysteresis law.

Hysteresis may switch at different spatial points at different time moments, dividing the spatial domain into subdomains where hysteresis has the same state and thus defining spatial topology of hysteresis. The boundaries between the subdomains are free boundaries whose motion depends both on the reaction-diffusion equation and hysteresis. The interplay of those two leads to formation of spatio-temporal patterns.

We formulate a theorem that states that the problem has a unique solution as long as this solution preserves the spatial topology of hysteresis, while the change of topology may occur only via a spatial nontransversality of the solution. In the end, we will present numerical results indicating a nontrivial behavior of the solution related to the change of topology.

# Asymptotic Properties of Solutions of Linearized Equations of Low Compressible Fluid Motion

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Consider a low compressible barotropic fluid with the following equation of state: $\varrho = \varrho_0 + \alpha(\mathrm{p} - \mathrm{p}_{\mathrm{ref}})$, where $\varrho$ is the density, p is the pressure, $\alpha > 0$ is a *compressibility coefficient* (factor), $\varrho_0 > 0$, and $\mathrm{p}_{\mathrm{ref}} = \mathrm{const}$. Let $D \subset \mathbf{R}^d$ be a bounded domain with a piecewise smooth boundary, $d \in \mathbf{N}$, $d \geqslant 2$, and $T > 0$. The linearization of the Navier–Stokes equations in the cylinder $D \times (0, T)$ near an arbitrary state with constant density ($\varrho = \varrho_0$) for such fluid can be written in the following form:

$$\begin{gathered} \rho_t - (\mathbf{b}, \nabla)\rho + c\rho + \operatorname{div} \mathbf{u} = \sigma, \\ \mathbf{u}_t + \nabla p = -A\mathbf{u} + \rho \mathbf{f} + \mathbf{s}, \\ \rho = \alpha p, \\ \text{where} \quad -A\mathbf{u} \equiv \nu \Delta \mathbf{u} + \kappa \nabla \operatorname{div} \mathbf{u} - (\mathbf{a}, \nabla)\mathbf{u} + M\mathbf{u}, \end{gathered} \tag{1}$$

$\mathbf{b}, \mathbf{u}, \mathbf{f}, \mathbf{s}, \mathbf{a} \colon \overline{D \times (0,T)} \to \mathbf{R}^d$ are vector fields, $\rho, c, \sigma, p \colon \overline{D \times (0,T)} \to \mathbf{R}$ are scalar fields, $M = M(x,t)$ is a scalar matrix of the size $d \times d$, $x \in \overline{D}$, $t \in [0,T]$, and $\nu > 0$ and $\kappa \geqslant 0$ are viscosity coefficients. The fields $\rho, \mathbf{u}$ and $p$ are the unknowns in the system (1).

Let $\mathbf{b}|_{\partial D} = 0$. Consider the following initial and boundary conditions for (1):

$$\mathbf{u}|_{t=0} = \mathbf{u}^\circ, \quad p|_{t=0} = p^\circ, \quad \mathbf{u}|_{\partial D} = 0, \tag{2}$$

where $\mathbf{u}^\circ \colon \overline{D} \to \mathbf{R}^d$ and $p^\circ \colon \overline{D} \to \mathbf{R}$.

We present sufficient conditions for existence and uniqueness of weak solutions to the initial–boundary value problem (1), (2). We also study the behaviour of the solutions of (1), (2) for $\alpha \to 0$: we prove that these solutions converge to the solution of the initial–boundary value problem for the corresponding linearized *incompressible* Navier–Stokes equations. In the case of the complete Navier–Stokes(–Fourier) system, similar convergence has been studied by E. Feireisl, P.-L. Lions, N. Masmoudi, E.G. Shifrin, and other authors. For the system (1), (2), we shall give some analogues of their results as well as sufficient conditions for *strong* convergence of the pressure. Our main results can be stated as follows:

(1) In the general case, the velocity field $\mathbf{u} = \mathbf{u}_\alpha$ converges *weakly*;

(2) If the initial condition $\mathbf{u}^\circ$ for the velocity is solenoidal, then $\mathbf{u}_\alpha$ converges *strongly* and $p = p_\alpha$ converges $*$-*weakly*.

(3) If, moreover, the initial condition for the pressure coincides with the initial value $q^\circ$ of the pressure $q$ in the incompressible fluid and, in addition, $\partial_t \int_D q\, dx = 0$, then $p_\alpha$ converges *strongly*.

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## On Domains Inaccessible to Solutions of Quasi-Linear Hyperbolic Equations with Parabolic Degeneracy

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Initial and characteristic conditions, which cause strong parabolic degeneracy of non-strictly hyperbolic quasilinear equations are discussed. There are considered cases when inaccessible to solutions domains bounded by curves of degeneracy are in areas of influence of perturbations.

## Nonlinear Equations with Delay and Liesegang Rings

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Consider the parabolic equation of the form $Lu = F(u)$, where $u(x, t)$ is the desired function, $t$ is the time, $x$ is the point of $n$-dimensional space, $L$ is a linear parabolic operator. The domain of the operator $F$ are functions $u(s)$ with $s < t$.

We investigate the problem, which describes the diffusion of substances for which there is an influx or decrease of new portions of the substance, depending on the density, reaching a previous maximum or minimum. We show that the oscillating process occurs and the resulting pattern of distribution of matter in the two-dimensional case is similar in appearance to the Liesegang rings.

## Small Stochastic Perturbations of Hamiltonian Flows: a PDE Approach

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We present a PDE approach to the study of averaging principles for (small) stochastic perturbations of Hamiltonian flows in 2D, which is based on a recent joint work with Takis Souganidis. Such problems were introduced by Freidlin and Wentzel and have been the subject of extensive study in the last decade. If the Hamiltonian flow has critical points, then the averaging principle exhibits complicated behavior. Asymptotically, the slow (averaged) motion has 1D character and takes place on a graph, and the question is to identify the limit motion in terms of PDE problems. In their original work Freidlin and Wentzell, using probabilistic techniques, considered perturbations by Brownian motions, while later Freidlin and Weber studied, combining probabilistic and analytic techniques based on hypoelliptic operators, a special degenerate case. Recently Sowers revisited the uniformly elliptic case and constructed what amounts to approximate correctors for the averaging problem. Our approach is based on PDE techniques and is applied to general degenerate elliptic operators.

# Transmission Problems for Reactive Flow and Transport-Multiscale Analysis of the Interactions of Solutes with a Solid Phase

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The modelling of reactive flows and transport in media consisting of multiple phases, e.g. of a fluid and a solid phase in a porous medium, is giving rise to many open problems for multiscale analysis, in particular, at the interfaces.

So far, the interactions of the solvent with the solid phase are too roughly approximated in many applications. In this lecture, we are discussing a more detailed mathematical representation of the processes in the solid on the micro-scale and are going to sketch the analysis of the arising transmission problems.

The following specific transmission conditions on the interface between the solid and the fluid phase are considered:

- the continuity of the fluxes of the solutes;
- the following nonlinear relation of the concentrations:
  $h(v_\varepsilon) = w_\varepsilon$.

Here $w_\varepsilon$ is the vector of concentrations in the solid phase and $v_\varepsilon$ represents the concentrations in the fluid phase. $\varepsilon$ is the scale parameter of the porous media. The structure of $h$ is determined referring to arguments from statistical physics.

The following two problems have to be solved:

(1) to investigate the existence and uniqueness of solutions for a fixed $\varepsilon$;

(2) to derive estimates of the solutions needed to pass to the scale limit $\varepsilon \to 0$ and to formulate effective equations.

Trying a standard weak formulation for the underlying partial differential equations does not work since the nonlinear relation on the interface cannot be integrated in functionals or in function spaces directly.

Whereas a scalar diffusion-reaction equation with this nonlinear transmission condition could be solved, the problem for systems was open up to now. Here, a relaxation approach is used to solve the transmission problem for systems for fixed scale $\varepsilon$, using structural assumptions on $h$. Finally, the scale limit is discussed and an effective system is derived.

The results obtained here are based on arguments used by Jäger and Kacur for relaxation approximations of nonlinear parabolic systems.

# On the Influence of Nonlinear Dissipative and Damping Terms for Hyperbolic Equations

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The initial and characteristic problems for wave equations with nonlinear dissipative and damping terms are considered. The uniqueness, local and global existence, and blow-up of solutions of the problems mentioned are investigated. The paper's originality is the coalescence of two standard methods: a priori estimates of solutions in the class of continuous functions is given by energetic methods; basing on this result, a priori estimates in the class of continuously differentiable functions is obtained by means of the classical method of characteristics.

# Existence of Very Weak Solutions for Nonlinear Elliptic Equations and Systems

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We consider high-order nonlinear elliptic system of type $\operatorname{div}^m A(x, D^m u) = \operatorname{div}^m f(x)$, $x \in \mathbb{R}^n$, with the natural energetic space $W_p^m$ and standard structure conditions (e.g. $m, p$-Laplacian). A term *very weak solution* means a solution in a space weaker than the natural energetic one. As for Lebesgue scale of spaces, a priori estimates of solutions in $W^m_{p-\varepsilon}$ are well known (J. Lewis (1993); T. Iwaniec and C. Sbordone (1994)), but existence results in $W^m_{p-\varepsilon}$ still absent due to the lack of the monotonicity in spaces different from $W_p^m$. The solvability is known only in grand Sobolev spaces $W^m_{p)}$ with the property $L_p \subset L_{p)} \subset L_{p-\varepsilon,loc}$ for any $\varepsilon > 0$.

We consider our system in the scale of dual Morrey spaces $W^m_{p,a} = \{u : D^m u \in L_{p,a}\}$, where $L_{p,a} = L_{p,a}(R^n)$, $1 < p < \infty$, $0 < a < n(p-1)$, is defined by the norm

$$||f||^p_{p,a} = \inf_{\sigma} \int_{R^n} |f|^p \omega dx\,, \qquad \omega(x) = \Big( \int_{R^{n+1}_+} r^{a/(1-p)} 1_{\{|x-y|<r\}} d\sigma(y,r) \Big)^{1-p},$$

and inf is taken over nonnegative Borel measures $\sigma$ on $R^{n+1}_+ = \{(y,r) : y \in R^n, r > 0\}$ with normalization $\sigma(R^{n+1}_+) = 1$.

Unlike spaces with Lebesgue exponent $q \neq p$, dual Morrey spaces allow us to establish the pseudomonotonicity of nonlinear operators for some range of $a$, which leads to the next result.

**Theorem.** *Let $p \in (1, n)$. Then there exists $a^* > 0$ depending on $n, p, m$, and the ellipticity modulus of the system such that for $a \in (0, a^*)$ and $|f|^{p'/p} \in L_{p,a}$ there exists a solution $u \in W^m_{p,a}$ such that*

$$||D^m u||_{p,a} \leqslant c\, ||\, |f|^{p'/p} ||_{p,a}\,.$$

Explicit estimates of $a^*$ are given for $p$ close to 2.

This is the first existence result for nonlinear elliptic systems with $p \neq 2$ in spaces, to say, considerably weaker than the natural energetic one.

## Periodic Solutions Bifurcation from the Cycle with Multidimensional Degeneracy for a Neutral Type Equation with a Small Delay

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For an equation of the form

$$\dot{x}(t) = f(x(t), x(t - \varepsilon h)) + a\dot{x}(t - \varepsilon h) + \varepsilon g(t, x(t), x(t - \varepsilon h), \varepsilon) + \varepsilon b(t)\dot{x}(t - \varepsilon h), \quad (1)$$

where $x : \mathbb{R} \to \mathbb{R}^n$, functions $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ are sufficiently smooth, $a$ and $b(t)$are $n \times n$-matrices with $\|a\| \leqslant q < 1$, and $h > 0$, the bifurcation problem of $T-$periodic solution from limit cycle is considered. Assume that Eq. (1) with $\varepsilon = 0$ has a $T-$periodic cycle and linearized on this cycle equation has adjoint Floquet solutions. Let the cycle of Eq. (1) be parametrized by parameter $\theta \in [0, T]$. The existence conditions for bifurcate solutions of the form

$$x^\varepsilon = x(\theta) + \varepsilon\mu_0 e_0(\theta) + \varepsilon^2 \sum_{j=1}^{m} \mu_j e_j(\theta) + \varepsilon y_0(\theta) + O(\varepsilon^3),$$

where $e_0(\theta)$ is the initial function for the limit cycle of the linearized equation and $e_j(\theta), j = 1, \ldots, m, m < n$, are initial functions for adjoint Floquet solutions are obtained. The coefficients $\mu_j, j = 0, \ldots, m$ and the function $y_0(\theta)$ can be found in explicit way.

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## On Nilmanifolds Admitting Anosov Diffeomorphisms

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After introducing Anosov diffeomorphisms, which are structurally stable and therefore being important for the dynamical systems, we will consider the following question raised by D. V. Anosov in the Moscower Conference: “What compact $M$ admits Anosov diffeomorphism?”

This classification problem remains unsolved. Nevertheless, there are some results with algebraic methods, which show that nilmanifolds $N/\Gamma$, where N is a nilpotent Lie group, $\Gamma$ is a uniform discrete subgroup of N, and infranilmanifolds admit Anosov diffeomorphisms. We will give some examples of nilmanifolds and a construction of a non-toral Anosov diffeomorphism.

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# Cesàro Convergence of Spherical Averages for Measure-Preserving Actions of Markov Semigroups


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The talk is based on a joint work with A. Bufetov and M. Khristoforov [1].

Consider a semigroup $\Gamma$ that is Markov with respect to a set of generators $O$. Recall the definition of a Markov semigroup. The semigroup $\Gamma$ has a natural norm: $|g|_O$ is a minimal length of a word representing $g$; let $S_O(n) = \{g : |g|_O = n\}$. Consider a directed graph $\mathbf{G}$, its vertex $v_0$ and a map $\xi\colon \mathcal{E}(\mathbf{G}) \to O$ from the set of its edges to the set $O$. This map is naturally extended to the map $\bar{\xi}$ from the set $\mathcal{P}(v_0)$ of all finite paths in $\mathbf{G}$ starting at $v_0$, to the group $\Gamma$: $\bar{\xi}(e_1 \dots e_n) = \xi(e_1)\dots\xi(e_n)$. The semigroup $\Gamma$ is called *Markov* with respect to the set $O$ if the map $\bar{\xi}$ is bijective and it maps any path of length $n$ into $S_O(n)$.

Suppose that the semigroup $\Gamma$ acts on a probability space $(X,\nu)$ by measure-preserving transformations $T_g$, $g \in \Gamma$. Take any function $\varphi \in L^1(X,\nu)$ and consider the sequence of its *spherical averages*:

$$s_n(\varphi) = \frac{1}{\# S_O(n)} \sum_{g\in S_O(n)} \varphi \circ T_g$$

($\#$ stands for the cardinality of a finite set; if $S_O(n) = \varnothing$, then we set $s_n(\varphi) = 0$). Next, consider the Cesàro averages of the spherical averages:

$$c_N(\varphi) = \frac{1}{N} \sum_{n=0}^{N-1} s_n(\varphi).$$

**Theorem.** *Let $\Gamma$ be a Markov semigroup with respect to a finite generating set $O$. Assume that $\Gamma$ acts by measure-preserving transformations on a probability space $(X,\nu)$. Then for any $p$, $1 \leqslant p < \infty$, and any $\varphi \in L^p(X,\nu)$ the sequence $c_N(\varphi)$ converges in $L^p(X,\nu)$ as $N \to \infty$. If, additionally, $\varphi \in L^\infty(X,\nu)$, then the sequence $c_N(\varphi)$ converges $\nu$-almost everywhere as $N \to \infty$.*

It was shown by Gromov [3] that any word hyperbolic group is Markov with respect to any symmetric set of generators. Thus, this theorem can be applied to any hyperbolic group.

In case of an irreducible graph $\mathbf{G}$, the theorem was proven earlier by Bufetov [2]. The proof in the general case is obtained through a decomposition of the graph $\mathbf{G}$ into smaller blocks (eventually irreducible ones).

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# The Center of Excellence G-RISC


N. Kolanovska

The German–Russian Interdisciplinary Science Center (G-RISC)


The German–Russian Interdisciplinary Science Center (G-RISC) is a Center of Excellence established in March 2010. The interdisciplinary center of excellence builds on a long tradition of scientific cooperation between scientists of Russia and Germany. The main offices are based at St. Petersburg State University and Freie Universität Berlin. G-RISC relies on funding and regulations of the German Academic Exchange Service (DAAD) and the German Federal Foreign Office. G-RISC provides a unique interdisciplinary research platform supporting education and research in binational projects between Russia and Germany. The focus is interdisciplinary researches in four important areas of natural sciences: physics, geophysics, physical chemistry, and mathematics.

The official launch of the Center of Excellence was preceded by a scientific competition for the best project proposals with 23 projects being awarded funding in the first term of 2010. This type of competition is organized biannually. The next call for proposals is scheduled for October 2011, where the deadline is set to October 31, 2011.

G-RISC primarily funds the mobility of young researchers between Russia and Germany. First of all, this concerns research stays in laboratories of the partner groups in the other country. For outstandingly bright Russian students who are involved in interdisciplinary research projects with German partner groups, it is also possible to become a sur-place stipend for maximum half a year. It is anticipated that research stays and stipend are increasing the chances for stable long-term Russian–German collaborations. Each single project is important helping to tie researchers and research interests together and to develop novel and interdisciplinary research between Russia and Germany.

In total, more than 30 institutions and more than 100 groups from Russia and Germany conduct researches and teach at the center.

In the presentation, I would like to describe how does G-RISC works, to explain the main aspects, for example, what is the administrative structure of Center, how to apply for a proposal, and what is funded by G-RISC. Moreover, some statistics regarding to the Center and the participating institutions would be presented.

# On Abstract Green's Identity for Sesquilinear Forms

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1°. Let for arbitrary Hilbert spaces $E$, $F$, and $G$ (with introduced scalar products) the following assumptions be fulfilled.

(i) The space $F$ is boundedly embedded in $E$, $F \hookrightarrow E$.

(ii) There exists an abstract trace operator $\gamma : F \to G$ and $\mathcal{R}(\gamma) =: G_+ \hookrightarrow G$.

(iii) There exists a sesquilinear form $\Phi(\eta, u)$, $\eta, u \in F$, such that $|\Phi(\eta, u)| \leqslant c_1 \|\eta\|_F \|u\|_F$, $\operatorname{Re}\Phi(u, u) \geqslant c_2 \|u\|_F^2$, $c_1 \geqslant c_2 > 0$.

**Theorem 1.** *If assumptions* (i)–(iii) *hold then the following Abstract Green's Identity is valid*:

$$\Phi(\eta, u) = \langle \eta, Lu \rangle_E + \langle \gamma\eta, \partial u \rangle_G, \quad \forall \eta, u \in F, \quad Lu \in F^*, \quad \partial u \in (G_+)^*. \tag{1}$$

Here $Lu$ is an abstract differential expression corresponding to the form $\Phi(\eta, u)$ and $\partial u$ is an abstract conormal derivative. They are defined uniquely by the data of the problem.

2°. Let for projections $p_k$, $k = \overline{1, q}$, acting in $G_+$, the following assumptions be fulfilled.

(iv) $p_k = \omega_k \rho_k$ (or $p_k^* = \rho_k^* \omega_k^*$) where $\rho_k$ is a bounded restriction operator acting from $G_+$ on $(G_+)_k := \rho_k G_+$, and $\omega_k$ is bounded extension operator acting from $(G_+)_k$ onto $p_k G_k$, $k = \overline{1, q}$.

(v) $\rho_k \omega_k = (I_+)_k$ (an identity operator on $(G_+)_k$), $k = \overline{1, q}$.

(vi) $\sum\limits_{k=1}^{q} p_k = I_+$ (an identity operator on $G_+$).

**Theorem 2.** *Under assumptions* (iv)–(vi)*, Abstract Green's Identity* (1) *has the form*

$$\begin{gathered}\Phi(\eta, u) = \langle \eta, Lu \rangle_E + \sum_{k=1}^{q} \langle \gamma_k \eta, \partial_k u \rangle_{G_k}, \quad \forall \eta, u \in F, \\ \gamma_k \eta := \rho_k \gamma \eta \in (G_+)_k, \quad \partial_k u := \omega_k^* \partial u \in (G_+)_k^*, \quad (G_+)_k \hookrightarrow G_k \hookrightarrow (G_+)_k^*, \quad k = \overline{1, q}.\end{gathered} \tag{2}$$

*Here $\gamma_k = \rho_k \gamma$ is an abstract trace operator on the $k$-th part of the boundary and $\partial_k = \omega_k^* \partial$ is the corresponding conormal derivative.*

3°. We consider some examples of applications of formulas (1) and (2) for classical boundary-value problems and for problems in hydrodynamics and elasticity theory.

# Dynamical Coherence Implies Central Shadowing

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Let $M$ be an $n$-dimensional $C^1$-smooth compact manifold, $\operatorname{dist}(\cdot,\cdot)$ be the Riemannian metrics on $M$, and $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. Consider the space $\operatorname{Diff}^1(M)$ of $C^1$-smooth diffeomorphisms $f : M \to M$.

**Definition 1.** A diffeomorphism $f \in \operatorname{Diff}^1(M)$ is called *partially hyperbolic* if it or its fixed power $f^m$ satisfies the following property. There exist a continuous bundle $TM = E^s \oplus E^u \oplus E^c$ and continuous functions $\nu, \hat{\nu}, \gamma, \hat{\gamma} : M \to (0, +\infty)$ such that $\nu, \hat{\nu} < 1$, $\nu < \gamma < \hat{\gamma} < \hat{\nu}^{-1}$ and for all $v \in \mathbb{R}^n$, $|v| = 1$, $x \in M$

$$|Df(x)v| \leqslant \nu(x) \quad \text{if} \quad v \in E^s(x); \qquad |Df(x)v| \leqslant \hat{\nu}^{-1}(x) \quad \text{if} \quad v \in E^u(x);$$

$$\gamma(x) \leqslant |Df(x)v| \leqslant \hat{\gamma}(x) \qquad \text{if} \quad v \in E^c(x).$$

**Definition 2.** We say that a $k$-dimensional distribution $E$ over $TM$ is *uniquely integrable* if there exists a $k$-dimensional foliation $W$ of the torus $M$, whose leaves are tangent to $E$ at every point. Also, any $C^1$-smooth path tangent to $E$ is embedded to a unique leaf of $W$.

**Definition 3** ([1])**.** A partially hyperbolic diffeomorphism $f$ is *dynamically coherent* if both distributions $E^{cs}$ and $E^{cu}$ are uniquely integrable.

Then, as it was proved in [2], both foliations $W^{cs}_{loc}$ and $W^{cu}_{loc}$ tangent to $E^{cs}$ and $E^{cu}$ respectively contain a subfoliation $W^c_{loc}$ tangent to $E^c$. We denote by $W^\tau_\varepsilon(x)$ the connected component of the set $W^\tau(x) \bigcap B(x, \varepsilon)$, which contains the point $x$.

**Definition 4.** A sequence $\{x_k : k \in \mathbb{Z}\}$ is called a *central d-pseudotrajectory* ($d > 0$) if $\operatorname{dist}(f(x_k), x_{k+1}) \leqslant d$ and $f(x_k) \in W^c_{loc}(x_{k+1})$ for all $k \in \mathbb{Z}$.

**Definition 5.** We say that the diffeomorphism $f$ satisfies the *Lipschitz central shadowing property* if there exists $L > 0$ such that for any $\varepsilon > 0$ and any $\varepsilon$-pseudotrajectory $\{x_k : k \in \mathbb{Z}\}$ there exists a central $L\varepsilon$ pseudotrajectory $y_k$ such that $\operatorname{dist}(x_k, y_k) \leqslant L\varepsilon$ for all $k \in \mathbb{Z}$.

We prove the analogue of the classical Anosov shadowing lemma for partially hyperbolic diffeomorphisms.

**Theorem 1.** *Let a diffeomorphism $f \in \operatorname{Diff}^1(M)$ be partially hyperbolic and satisfy the dynamical coherence property. Then $f$ satisfies the Lipschitz central shadowing property.*

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# Some Properties of the $n$-th Order Operator Pencil


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Let $X$ and $Y$ be Banach spaces. We consider the $n$-th order pencil

$$\lambda \mapsto \lambda^n F_n + \lambda^{n-1} F_{n-1} + \cdots + \lambda F_1 + F_0,$$

where $F_k\colon X \to Y$, $k = 0, 1, \ldots, n$, are bounded linear operators acting from $X$ to $Y$. The *resolvent set* $\rho(\{F_k\})$ of the pencil is the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda^n F_n + \lambda^{n-1} F_{n-1} + \cdots + \lambda F_1 + F_0$ is invertible; the complement $\sigma(\{F_k\}) = \mathbb{C} \setminus \rho(\{F_k\})$ is called the *spectrum* of the pencil. The family $R_\lambda = (\lambda^n F_n + \lambda^{n-1} F_{n-1} + \cdots + \lambda F_1 + F_0)^{-1}$, $\lambda \in \rho(\{F_k\})$, is called the *resolvent*.

We denote by $\mathcal{BO}(Y, X)$ the Banach space of all bounded linear operators acting from $X$ to $Y$. We denote by $\mathcal{BO}_{(\{F_k\})}(Y, X)$ the closure with respect to the norm of the space $\mathcal{BO}(Y, X) \oplus \cdots \oplus \mathcal{BO}(Y, X)$ of the linear span of all *vector resolvent* $\mathfrak{R}_\lambda = (R_\lambda, \lambda R_\lambda, \lambda^2 R_\lambda, \ldots, \lambda^{n-1} R_\lambda)$. We endow $\mathcal{BO}_{(\{F_k\})}(Y, X)$ with the multiplication

$$(A_1, \ldots, A_n) \circledast (B_1, \ldots, B_n) = (C_1, \ldots, C_n),$$

where $C_k$ are defined by the formula

$$C_{k+1} = \sum_{i=0}^{n-k-1} \sum_{l=1}^{n-k-i} A_{k+l} F_{n-i} B_{n+1-i-l} - \sum_{i=n-k+1}^{n} \sum_{l=0}^{i-(n-k+1)} A_{k+l} F_{n-i} B_{n+1-i+l}.$$

The family $\mathfrak{R}_\lambda$, $\lambda \in \rho(\{F_k\})$, satisfies the $\circledast$-*Gilbert identity*

$$\mathfrak{R}_\lambda - \mathfrak{R}_\mu = -(\lambda - \mu) \mathfrak{R}_\lambda \circledast \mathfrak{R}_\mu.$$

The space $\mathcal{BO}_{(\{F_k\})}(Y, X)$ is a commutative Banach algebra with respect to the multiplication $\circledast$. This algebra has a unit if and only if the operator $F_n$ is invertible. In this case, the unit is the element $(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, F_n^{-1})$.

Let us denote by the symbol $\mathcal{O}\big(\sigma(\{F_n\})\big)$ the set of all functions that are analytic in some neighborhood of the set $\sigma(\{F_n\})$. Obviously, $\mathcal{O}\big(\sigma(\{F_n\})\big)$ is an algebra with unit with respect to the pointwise operations.

**Theorem.** *Let the operator $F_n$ be invertible. Then the mapping*

$$\Upsilon = \chi_0 \oplus \chi_1 \oplus \chi_2 \oplus \cdots \oplus \chi_{n-1}\colon \mathcal{O}\big(\sigma(\{F_n\})\big) \to \mathcal{BO}_{(\{F_n\})}(Y, X),$$

*where the mappings $\chi_k\colon \mathcal{O}\big(\sigma(\{F_n\})\big) \to \mathcal{BO}(Y, X)$, $k = 0, \ldots, n-1$, are defined by the formulas*

$$\chi_k(f) = \frac{1}{2\pi i} \int_\Gamma \lambda^k f(\lambda)(\lambda^2 E + \lambda F + H)^{-1}\, d\lambda,$$

*and $\Gamma$ surrounds the spectrum $\sigma(\{F_k\})$, is a morphism of algebras with unit.*

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# On Two-Parameter Analogue of Malkin's Theorem

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We consider the system

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1) + \mu_1\varphi_1(x_2) \\ \dfrac{dx_2}{dt} = f_2(x_2) + \mu_2\varphi_2(x_1), \end{cases} \tag{1}$$

where $f_1 : R^n \to R^n$, $f_2 : R^m \to R^m$, $\varphi_1 : R^m \to R^n$, $\varphi_2 : R^n \to R^m$, and $f_1, f_2, \varphi_1, \varphi_2$ are continuously differentiable and $\mu_1, \mu_2$ are small positive parameters. It's supposed that the limit equations $\dot{x_1} = f_1(x_1)$ and $\dot{x_2} = f_2(x_2)$ have a $T_1$-periodic solution $x_1 = \psi_1(t)$ and $T_2$-periodic solution $x_2 = \psi_2(t)$. We assume that 1 is a simple eigenvalue of the translation operators by the trajectories of the equations

$$\dot{y_1} = f_1^{'}(\psi_1(t))y_1, \tag{2}$$

$$\dot{y_2} = f_2^{'}(\psi_2(t))y_2 \tag{3}$$

on the times $T_1$ and $T_2$ respectively.

Let $\xi_1$ and $\xi_2$ be non-trivial periodic solutions of the adjoint to Eqs. (2) and (3) and the scalar products $< \varphi_1(\psi_2(\cdot)), \xi_1(\cdot) >$ and $< \varphi_2(\psi_1(\cdot)), \xi_2(\cdot) >$ have bounded primitives.

The last assumption is the following: for the functions

$$P_1(h_1) = \lim_{n\to\infty} \frac{1}{t}\int_0^t < \varphi_1(\psi_2(s+h_1)), \xi_1(s) > ds,$$

$$P_2(h_2) = \lim_{n\to\infty} \frac{1}{t}\int_0^t < \varphi_2(\psi_1(s+h_2)), \xi_2(s) > ds,$$

the derivatives $\frac{dP_1}{dh_1}, \frac{dP_2}{dh_2}$ are different from zero.

**Theorem.** *Let $c_1, c_2 > 0$, and $0 < \varepsilon < \frac{3}{4}$ be constants and $\mu_1$ and $\mu_2$ belong to the set described by the inequalities $c_1\mu_1^{(2-\varepsilon)} < \mu_2 < c_2\mu_1^{(1/2+\varepsilon)}$. Let the above conditions be satisfied. Then system* (1) *has a unique almost periodic solution for all sufficiently small values of parameters $\mu_1$ and $\mu_2$.*

# Kohno Systems on Manin–Schechtman Configuration Spaces

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It is well known that the Frobenius condition of the integrability of the formal KZ equation

$$\mathrm{d}\, y(z) = \left( \sum_{1 \leqslant i < j \leqslant n} X_{ij} \frac{\mathrm{d}(z_i - z_j)}{z_i - z_j} \right) y(z). \tag{1}$$

on the complex linear space $\mathbb{C}^n$ is equivalent to relations in Lie–Chen–Kohno algebra $\mathcal{L}_K$ [3] that generated by symbols $X_{ij}$, $i \neq j$, $X_{ij} = X_{ji}$ and the relations

$$[X_{ij},\, X_{ik} + X_{jk}] = 0, \;\; 1 \leqslant i < j < k \leqslant n, \tag{$*$}$$

$$[X_{ij},\, X_{kl}] = 0, \;\; \{i, j\} \cap \{k, l\} = \varnothing. \tag{$**$}$$

We consider a Lie subalgebra $\mathcal{L}_{MS}$ in $\mathcal{L}_K$, generated by elements $X_J = \sum_{i<j,\, i,j \in J} X_{ij}$, where $J$ is a subset of the set $\{1, 2, \ldots, n\}$. Let $|J|$ be the number of elements in the $J$. For $X_J$, the following relations are fulfilled:

$$[X_J,\, \sum_{I \subset K} X_I] = 0, \;\; \forall K \subset \{1, 2, \ldots, n\}, \; |K| = k+2, \; I \subset K, \; J \subset K, \; |I| = |J| = k+1. \tag{$***$}$$

These relations are are a part of Frobenius relations for the integrability of the Kohno system [4]

$$\mathrm{d}\, y(z) = \left( \sum_{J,\, |J| = k+1} X_J \frac{\mathrm{d}\, \varphi_J(z_1, \ldots, z_n)}{\varphi_J(z_1, \ldots, z_n)} \right) y(z) \tag{2}$$

on the Manin–Schechtman space [6] of configurations of $n > k$ hyperplanes in $\mathbb{C}^k$, where

$$X_J = \sum_{i<j,\, i,j \in J} X_{ij}, \;\; |J| = k+1.$$

Functions $\varphi_J(z_1, \ldots, z_n)$ are some linear forms on $\mathbb{C}^n$. If $n = k+2$, then relations (***) are sufficient for the integrability of (2). For $n = k+2$, we study the Kohno systems obtained from integrable Jordan–Pochhammer systems [2, 3]

$$\mathrm{d}\, y(z) = \left( \sum_{1 \leqslant i < j \leqslant n} A_{ij}(\lambda) \frac{\mathrm{d}(z_i - z_j)}{z_i - z_j} \right) y(z), \tag{3}$$

where each matrix $A_{ij}(\lambda)$ has the size $n \times n$ and with non-zero entries only places $(i,\, i)$, $(i,\, j)$, $(j,\, i)$ and $(j,\, j)$ equals to complex numbers $\lambda_j$, $-\lambda_i$, $-\lambda_j$ and $\lambda_i$ correspondingly that are taken from an ordered collection complex numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. We describe monodromy properties obtained systems. Some applications of these systems for the computation of volumes of non-Euclidean polyhedrons are considered [1, 5].

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# Hidden Oscillations in Dynamical Systems


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The problem of hidden oscillations in nonlinear control systems forces to develop new approaches of nonlinear oscillation theory. During initial establishment and development of theory of nonlinear oscillations in the first half of 20th century, the main attention has been given to the analysis and synthesis of oscillating systems for which the solution of existence problems of oscillating regimes was not too difficult. For many systems, the structure was such that they had oscillating solutions, the existence of which was "almost obvious". The arising of periodic solutions in these systems was well seen by numerical analysis when numerical integration procedure of the trajectories allowed one to pass from a small neighborhood of the equilibrium to a periodic trajectory. Therefore, the main attention of researchers was concentrated on the analysis of forms and properties of these oscillations (the "almost" harmonic, relaxation, synchronous, circular, orbitally stable ones, and so on).

Further, so called *hidden oscillations* came to light. They are oscillations, the existence of which is not obvious (they are "small" and, therefore, are difficult for numerical analysis or are not "connected" with equilibrium, i.e. the creation of a numerical integration procedure of trajectories for the passage from the equilibrium to a periodic solution is impossible). In the midpoint of twentieth century M. A. Aizerman and R. E. Kalman formulated two conjectures, which occupy, at once, attention of many famous scholars.

Similar situation is in attractors localization. The classical attractors of Lorenz, Rossler, Chua, Chen, and other widely-known attractors are those excited from unstable equilibria. From the computational point of view, this allows one to use numerical method, in which (after a transient) a trajectory started from a point of an unstable manifold in a neighborhood of the equilibrium reaches an attractor and identifies it. However, there are attractors of another type: *hidden attractors, which are a basin of attraction of which does not contain neighborhoods of equilibria.*

In this presentation, the application of special analytical-numerical algorithms for hidden oscillations and hidden attractor localization are discussed. A construction of counterexamples for Aizerman's and Kalman's conjectures and existence of hidden attractor in Chua's systems are demonstrated.

## A Model of Collinear Tri-Atomic Chemical Reactions: Billiard in the Angle with Potential

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A geometrical model which captures the main ingredients governing tri-atomic co-linear chemical reactions is proposed. This model is neither near-integrable nor hyperbolic, yet it is, but it is still possible to analyze it, using a combination of the recently developed tools for systems with steep potentials and with linear theory near a center-saddle equilibrium. Thus, the non-trivial dependence of the reaction rates on parameters, initial conditions, and energy is explained. Conditions under which the phase space transition state theory assumptions are satisfied and conditions under which these fail are derived. Mathematically, this model is a classical Hamiltonian system with a quadratic potential with a saddle critical point defined on the configuration space being an angle. The motion of the system follows this Hamiltonian system up to the moment when it hits the boundary walls of the angle, then it performs a jump by the billiard law.

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The talk is based on the joint paper with V. Rom-Kedar (The Weizmann Institute of Science, Israel).

## Bifurcation without Parameters

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We study dynamical systems with manifolds of equilibria near points at which normal hyperbolicity of these manifolds is violated. Manifolds of equilibria appear frequently in classical bifurcation theory by continuation of a trivial equilibrium. Here, however, we are interested in manifolds of equilibria which are not caused by additional parameters. In fact, we require the absence of any flow-invariant foliation transverse to the manifold of equilibria at the singularity. Therefore, we call the emerging theory bifurcation without parameters.

Albeit the apparent degeneracy of our setting (of infinite codimension in the space of all smooth vector fields), there is a surprisingly rich and diverse set of applications ranging from networks of coupled oscillators, viscous and inviscid profiles of stiff hyperbolic balance laws, standing waves in fluids, binary oscillations in numerical discretization, population dynamics, cosmological models, and many more.

In this lecture we will give an overview of the behavior of flows near bifurcation points without parameters and discuss new results on bifurcations of higher codimension.

## Subordinated Conditions for a Tensor Product of Two Minimal Differential Operators

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In this communication, we describe the linear space $L(P)$ of minimal differential polynomials $Q(D)$ subordinated in the $L^\infty(\mathbb{R}^n)$ norm to the tensor product

$$P(D) = P_1(D) \otimes P_2(D) := P_1(D_1, \dots, D_{p_1}, 0, \dots, 0) \cdot P_2(0, \dots, 0, D_{p_1+1}, \dots, D_n)$$

of two elliptic operators $P_1(D)$ and $P_2(D)$ acting on different variables.

We prove that if $P_1(\xi)$ and $P_2(\xi)$ are homogeneous symbols, then the space $L(P)$ is minimal possible, i.e., the inclusion $Q \in L(P)$ is equivalent to the equality $Q(D) = c_1 + c_2 P(D)$ (see [1,2]).

We also consider the case of the product $P(D_1, D_2) = p_1(D_1)p_2(D_2)$ of two ordinary differential operators. We show that if all the zeros of the symbol $p_1(\xi_1)$ are real and simple, the dimension of the space $L(P)$ depends on the number of real zeros of the symbol $p_2(\xi_2)$.

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## Periodic and Homoclinic Travelling Waves on Lattices

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We consider the following one-dimensional system of infinitely many ODEs:

$$\ddot{q}_j + f'(q_j) = V'(q_{j+1} - q_j) - V'(q_j - q_{j-1}), \quad j \in \mathbf{Z}.$$

We investigate the existence of travelling wave solutions, i.e., solutions of the type

$$q_j(t) = u(j - ct),$$

where $c > 0$ is a real constant and $u$ is a real-valued function. Then the problem is reduced to the solving of the following forward-backward differential equation:

$$c^2 u''(s) + f'(u(s)) = V'(u(s+1) - u(s)) - V'(u(s) - u(s-1)). \qquad (1)$$

$T$-periodic solutions of (1) are the critical point of a functional $\Phi_T$. Under very natural assumptions on $f$ and $V$, non-constant periodic solutions are found by means of the mountain pass or linking theorem. Solutions of (1) homoclinic to 0 correspond to the critical points of a functional $\Phi_\infty$, which, unfortunately, does not satisfy the so-called Palais–Smale condition. We find a non-trivial critical point $u_\infty$ of $\Phi_\infty$ as a limit of some sequence $\{u_k\}_k$, where $u_k$ is a critical point of $\Phi_k$.

## Mathematical Modelling and Simulation of the Swelling of the Brain Cell under Ischaemic Conditions

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In this contribution, we are presenting our results on the mathematical modelling of the brain cell swelling under pathological conditions during ischaemic brain infarct. The swelling is a result of the osmotic transport of water across the cellular membrane caused by the the ionic concentration difference between extra- and intracellular spaces.

The cell and the surrounding membrane are modelled as deformable porous media and are described by the Biot poroelasticity equations: a system of effective equations for the flow velocity (Darcy's law) and the deformation of the structure (linearized elasticity equations). This system is coupled with the Navier–Stokes equations for the extracellular fluid. Providing a transmission condition at the interface between the porous medium and the free fluid flow is, in general, an open question. The use of appropriate assumptions allows us to formulate suitable transmission conditions, taking into account special features of the problem.

Numerical results for the reduced model are obtained by using software DUNE in collaboration with the Scientific Computing Group of IWR, University of Heidelberg.

## Implicit Difference Schemes for Quasilinear Parabolic Functional Equations

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We present a new class of numerical methods for quasilinear parabolic functional differential equations with initial boundary conditions of the Robin type. The numerical methods are difference schemes which are implicit with respect to the time variable. We give a complete convergence analysis for the methods and we show that the new methods are considerably better than the explicit schemes. The proof of the stability is based on a comparison technique with nonlinear estimates of the Perron type for given functions with respect to functional variables. Results obtained in the paper can be applied to differential equations with deviated variables and to differential integral problems.

## Variational Gaussian Approximation in the Fluctuating Field Theory

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We consider the problem of the calculating of the partition function

$$Z = \int \exp(-F(V)/T)\, DV, \tag{1}$$

which is given by the functional integral over an external field $V_j(\tau)$ that fluctuates in a space $j$ and in "time" $\tau \in [0, 1/T]$, where $T$ is the temperature.

Practical calculation of integral (1) requires an approximation of the fluctuating field $V$. The simplest approximation is obtained by the saddle-point method, which replaces the fluctuating field by its mean value $\bar{V}$. However, the mean-field approximation is insufficient for a quantitative description of cooperative phenomena such as magnetism.

Based on the free energy minimum principle, we develop a method to calculate functional integrals (1) with the help of the Gaussian approximation $F^{(2)}(V) = \mathrm{Tr}(\Delta V^\dagger A \Delta V)$ that takes into account "dynamics" (quantum effects) and nonlocality of the fluctuations [1].

**Theorem.** *The mean and covariance matrix of the variational Gaussian approximation are calculated self-consistently from the system of nonlinear integral equations*

$$\left\langle \frac{\partial F(V)}{\partial V} \right\rangle_{(2)} = 0, \qquad A = \frac{1}{2} \left\langle \frac{\partial^2 F(V)}{\partial V^2} \right\rangle_{(2)}, \tag{2}$$

*where the average* $\langle \cdots \rangle_{(2)}$ *is calculated with the Gaussian probability density function proportional to* $\exp(-F^{(2)}(V)/T)$.

In the spin-fluctuation theory, the system of equations (2) can have several solutions below the temperature of the phase transition, which leads to a hysteresis in the temperature dependence [2]. The renormalization due to higher-order terms of the free energy $F(V)$ suppresses the critical fluctuations and yields a proper second-order phase transition [3].

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# Ginzburg–Landau Energy with Prescribed Degrees

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We consider the simplified Ginzburg–Landau energy $\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{(4\varepsilon)^2}\int_\Omega (1 - |u|^2)^2$. Here, $\Omega$ is a domain in $\mathbb{R}^2$ and $u$ is complex-valued. On $\partial\Omega$, we prescribe $|u| = 1$ and the winding numbers of $u$. This is one of the simplest models of critical equation leading to non-scalar bubbles. I will discuss existence/nonexistence results for minimizers/critical points. The talk is based on results of Berlyand, Dos Santos, Farina, Golovaty, Rybalko, and the lecturer.

# Asymptotic Properties of Schur–Weyl Duality

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Vershik and Kerov in 1985 gave asymptotic bounds for the maximal and typical dimensions of the irreducible representations of the symmetric group. It was conjectured by Grigori Olshanski that the maximal and typical dimensions of the isotypic components of the representations in Schur–Weyl duality accept similar asymptotic bounds. The isotypic components of this representation are parametrized by certain Young diagrams, and the relative dimensions of these components give rise to a measure on Young diagrams, which we call the Schur–Weyl measure. Philippe Biane in 2001 found the limit shape of a typical Young diagram with respect to the Schur–Weyl measure. We will discuss a proof of the conjecture which is based on showing that the limit shape found by Biane is the unique solution to a variational problem.

# Asymptotic Representations of Solutions of Elliptic Boundary-Value Problems in the Vicinity of Coefficient Discontinuity Line

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It is well known that presence of salient points in the boundary can lead to singularities of solutions of boundary-value problems and to difficulties in the use of numerical methods [1, 2]. One of the ways to get over the problem is to find the asymptotic representation of the solution in the vicinity of the irregularity of the boundary [3, 4] and then apply the mixed finite element method with singular test functions.

Elliptic boundary-value problem in a plane for the unknown function $u$ and discontinuous coefficient $\varepsilon$ equal to $\varepsilon_1$ in $D_1$ and $\varepsilon_2$ in $D_2$ is considered. There are conjugation conditions on the discontinuity line $C$ of the coefficient $\varepsilon$:

$$\begin{cases} \Delta u = f_1(M), & M \in D_1, \\ \Delta u = f_2(M), & M \in D_2, \end{cases} \qquad [u]|_C = 0, \qquad \left[\varepsilon \frac{\partial u}{\partial n}\right]\bigg|_C = 0, \tag{1}$$

where the discontinuity line $C$ of the coefficient $\varepsilon$ corresponds to rays $C_1$ и $C_2$ outgoing from the origin and to the angular value $\omega_0$. Let us assume that $f$ is a function from $V_{\gamma_1}^{l_1} \cap V_{\gamma_2}^{l_2}$, where $V_\gamma^l$ is the functional space with norm

$$\|u\|_{V_\gamma^l}^2 = \sum_{j+k \leqslant l} \left[ \int_0^{\omega_o} d\varphi \int_0^\infty r^{2(\gamma-l+j)} \left| \frac{\partial^{j+k} u}{\partial r^j \partial \varphi^k} \right|^2 r dr + \int_{\omega_o}^{2\pi} d\varphi \int_0^\infty r^{2(\gamma-l+j)} \left| \frac{\partial^{j+k} u}{\partial r^j \partial \varphi^k} \right|^2 r dr \right],$$

where $l \geqslant 0$ is integer and $\gamma$ is real.

The following representation of the solution have been obtained:

$$u(r,\varphi) = \sum_{h_1 < \nu_k^{(1)} < h_2} C_k^{(1)} r^{\nu_k^{(1)}} \Phi_k^{(1)}(\varphi) + \sum_{h_1 < \nu_k^{(2)} < h_2} C_k^{(2)} r^{\nu_k^{(2)}} \Phi_k^{(2)}(\varphi) + \Re(r,\varphi),$$

where $h_j = -\gamma_j + l_j + 1$, $\nu_k^{(j)}$, $C_k^{(j)}$, $j = 1, 2$, are constants and $\Phi_k^{(j)}(\varphi)$ are functions dependent on the angle variable only. There is an estimate for the smooth part of the solution: $\|\Re(r,\varphi)\|_{V_{\gamma_2}^{l_2+1}} \leqslant C \|f(r,\varphi)\|_{V_{\gamma_2}^{l_2}}$.

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# On Regular Solutions of the Cauchy Problem for Abstract Parabolic Equations

A. B. Muravnik

For the Cauchy problem

$$\frac{\partial u}{\partial t} = Lu, \ x \in (-\infty, +\infty), t > 0, \tag{1}$$

$$u(x, 0) = u_0(x), \ x \in (-\infty, +\infty), \tag{2}$$

where $L$ is a Fourier multiplier with even symbol $a(\xi)$, the solvability in classes of generalized functions is well known (see, e.g., [1]).

Assuming that $u_0$ is bounded and continuous and there exist $C, C_1, C_2 \in (0, +\infty)$ and $\alpha, \alpha_1, \alpha_2 \in (1, +\infty)$ such that

$$a(\xi) \leqslant \ln \frac{C}{1+|\xi|^\alpha}, \left|a'(\xi)e^{a(\xi)t}\right| \leqslant \frac{C_1}{1+|\xi|_1^\alpha}, \left|a''(\xi)e^{a(\xi)t}\right| \leqslant \frac{C_2}{1+|\xi|_2^\alpha}, \tag{3}$$

we prove that the solution is continuous in $(-\infty, +\infty)$ for any positive $t$.

To do that, we investigate the fundamental solution and, using the Wiener Tauberian theorem, show that it decays at infinity sufficiently fast under assumption (3).

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# Concentration along Submanifolds for the Problem $-\Delta u + \lambda u = u^{\frac{n-k+2}{n-k-2}}$ with Neumann Boundary Condition in Bounded Domains

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In this talk, we consider the equation $-\Delta u + \lambda u - u^{\frac{n-k+2}{n-k-2}} = 0$ in $\Omega \subset \mathbb{R}^n$ under zero Neumann boundary conditions, where $\Omega$ is open, smooth, and bounded, while $\lambda$ is a positive and large real number. We prove the existence of positive solutions concentrating along a submanifold $K$ of the boundary $\partial\Omega$ with dim $(K) = k$, as $\lambda \to +\infty$. This is a joint work with M. del Pino and F. Mahmoudi.

# On the Existence of Extremal Functions in the Maz'ya–Sobolev Inequality

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Denote by $x = (y; z) = (y_1, y'; z)$ a point in $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$, $n \geqslant 3$, $2 \leqslant m \leqslant n-1$. By $\mathcal{P}$ we denote the subspace $\{x \in \mathbb{R}^n : \ y = 0\}$.

Let $\Omega$ be a domain in $\mathbb{R}^n$. By $\mathcal{C}_0^\infty(\Omega)$ we denote the set of smooth functions with compact support in $\Omega$. For $1 \leqslant p < \infty$, we denote by $\dot{W}_p^1(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\nabla v\|_{p,\Omega}$. Obviously, $\dot{W}_p^1(\Omega) = \overset{o}{W}{}_p^1(\Omega)$ for bounded domains.

By definition, for $0 \leqslant \sigma \leqslant \min\{1, \frac{n}{p}\}$, we put $p_\sigma^* = \frac{np}{n-\sigma p}$. We discuss the attainability of the sharp constant in the so-called *Maz'ya–Sobolev inequality*

$$\||y|^{\sigma-1} v\|_{p_\sigma^*,\Omega} \leqslant \mathcal{N}(p,\sigma,\Omega) \cdot \|\nabla v\|_{p,\Omega}, \tag{1}$$

which holds true for any $v \in \dot{W}_p^1(\Omega)$ provided that

$$\begin{array}{lll} (a)\ \Omega \text{ is any domain in } \mathbb{R}^n & \text{for} & \frac{n(p-m)}{p(n-m)} < \sigma \leqslant 1; \\ (b)\ \Omega \subset \mathbb{R}^n \setminus \mathcal{P} & \text{for} & p > m,\ \sigma \leqslant \min\{\frac{n(p-m)}{p(n-m)}; \frac{n}{p}\},\ \sigma \neq 1; \\ (c)\ \Omega \subset \mathbb{R}^n \setminus (\ell \times \mathbb{R}^{n-m}) & \text{for} & p = m,\ \sigma = 0 \end{array} \tag{2}$$

(here $\ell$ is a ray in $\mathbb{R}^m$ beginning at the origin). Note that the case $p < n$, $\sigma = 1$, gives a conventional Sobolev inequality.

It is easy to see that for $p < n$ and $0 < \sigma < 1$, the sharp constant in (1) does not depend on $\Omega$ and *is not attained* for any $\Omega$ provided that $\Omega \cap \mathcal{P} \neq \varnothing$ and $\dot{W}_p^1(\Omega) \neq \dot{W}_p^1(\mathbb{R}^n)$.

We consider considerably more complicated case $\Omega \cap \mathcal{P} = \varnothing$, $\partial\Omega \cap \mathcal{P} \neq \varnothing$. First, we analyze the attainability of the sharp constant in (1) for $\Omega$ being a wedge $\mathcal{K} = K \times \mathbb{R}^{n-m}$ (here $K$ is an open cone in $\mathbb{R}^m$) or a "perturbed" wedge. Here we consider all $1 < p < \infty$ and $0 \leqslant \sigma < \min\{1, \frac{n}{p}\}$. Naturally, we suppose that $\Omega$ satisfies (2).

In the second part of the talk, we prove the attainability of the sharp constant in (1) in a bounded domain for $p = 2$ and $0 < \sigma < 1$. Note that our requirements on $\partial\Omega$ are considerably weakened comparing with the recent paper of Ghoussoub and Robert.

**Remark.** For $m = 1$, our problem of interest degenerates in a sense. On the another hand, the problem for $m = n$ corresponding to the Hardy–Sobolev inequality was investigated in a number of papers (see the survey [1], where the history of related problems and extensive bibliography was given).

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# Homogenization Limits of a Model System for Interaction of Flow, Chemical Reactions, and Mechanics in Cell Tissue

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Experimental researches are providing increasing information on biophysical and biochemical processes in cells and tissue. This information on cellular level has to be included in the mathematical modelling of the dynamics of biological tissue. Describing flow, transport and reactions of substances in and their interactions with mechanics of solid structures on a cellular level leads to a coupled system of nonlinear partial differential equations in complex geometric structures. The existence and uniqueness of the solution to the microscopic system is given in [1].

Using experimental information, the relevant parameters of the microscopic system have been determined in order to pass to a macroscopic scale limit. In the limit, when the scale parameter goes to zero, we obtain the quasi-static Biot system coupled with the upscaled reactive flow. Effective Biot's coefficients depend on the reactant concentration. Additionally to the weak two-scale convergence results, we prove the convergence of the elastic and viscous energies. These results are obtained in [2].

In the talk, we will present important aspects on this topic.

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# On Solvability of Boundary-Value Problems for Strongly Elliptic Differential-Difference Equations in Hölder Spaces with Translations in the Arguments of Low-Order Terms

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Let $Q \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial Q \in C^\infty$. We introduce the operator $R_Q = P_Q R I_Q$, where $I_Q$ is the operator of extension by zero in $\mathbb{R}^n \setminus Q$, $P_Q$ is the operator of restriction to $Q$, and the operator $R$ is defined as follows:

$$Ru(x) = \sum_{h \in \mathcal{M}} a_h u(x+h).$$

Here $\mathcal{M}$ is a finite set of vectors $h \in \mathbb{R}^n$ with integer coordinates, while $a_h$ are complex numbers.

We consider the following problem:

$$-\Delta u(x) + R_Q u(x) = f(x) \qquad (x \in Q) \tag{1}$$

with the homogeneous Dirichlet condition

$$u\big|_{\partial Q} = 0, \tag{2}$$

where $f(x) \in C^\sigma(\overline{Q})$, $(0 < \sigma < 1)$.

Assuming that operator $R_Q + R_Q^*$ is positive, we prove the existence and uniqueness of a classical solution $u \in C^{2+\sigma}(\overline{Q})$ of problem (1), (2).

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## Asymptotics of the Einstein–Vlasov System with Bianchi II Symmetry


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The late-time behaviour for the Einstein equation, which describes a gravitational field coupled to the Euler equations, where the matter is described by a perfect fluid, is well understood for spacetimes with a three-dimensional Lie group symmetry. My goal is to extend these results to some cases of the Einstein–Vlasov system, where the matter is modelled by a collisionless gas instead of a perfect fluid. In the talk, I will present advances in the case of the Heisenberg group (the Bianchi type II).

## Homogenization of the Discrete Diffusion-Absorption Equation


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Consider the diffusion process in the interval $(-R, R)$ containing a chain of "cells" (points) absorbing some substance. The integral quantity of the absorbed substance modifies the Young modulus of the chain. It means that the same force $\Phi$ applied to the chain generates smaller displacements of the "saturated cells" than in the case of the non-saturated structure. Assuming that the distance between neighboring cells in the equilibrium is a small positive parameter, we construct a continuous asymptotic approximation. We prove the estimate between the exact and asymptotic solution, previously justifying their existence.

1. For a given $R > 0$, consider the equation

$$u''(x) = \alpha h \sum_{j=0}^{K} \delta(x - Y_j^u) u(x) + f(x), \quad x \in (-R, R), \quad u'(-R) = 0, u(R) = 0,$$

where $f \in C([-R, R])$, $\alpha > 0, h > 0$, $Y_j^u = jhE_0(1 + qF[u])^{-1}$, $j = 0, ..., K$, and $F[u] = \int_0^{E_0} u(x)dx$. Here $0 < E_0 < R/2$, $q > 0$, $K$ is integer, and $h = 1/K$.

For sufficiently small $\alpha$ and $q$, the existence of a solution is proved.

2. Consider the homogenized problem

$$\hat{u}''(x) = \alpha \frac{1 + qF[\hat{u}]}{E_0} \chi_{[0, Y_K^{\hat{u}}]}(x) \hat{u}(x) + f(x), \quad x \in (-R, R), \quad \hat{u}'(-R) = 0, \hat{u}(R) = 0.$$

For sufficiently small $\alpha$, the existence and uniqueness of a solution is proved. The following estimate holds:

$$\|u - \hat{u}\|_C = O(h).$$

# Critical Points of a Family of Functionals Implied by a Stable Critical Point of a Single Limit

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Consider a family of functionals $J_\lambda$, $\lambda \geqslant 1$, a single limit $J_0$, and a stable critical point $\bar{u}$ of $J_0$. Then, for $\lambda \geqslant 1$, we get large critical points $u_\lambda$ of $J_\lambda$ such that $\|u_\lambda - \bar{u}\|_\lambda \to 0$ for $\lambda \to \infty$. The family of functionals $J_\lambda$ is given by

$$J_\lambda : H \to \mathbb{R}, \quad J_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 - k(u).$$

Here $H$ is a real Hilbert space and the norms $\|\cdot\|_\lambda$ are induced by a family of scalar products $\langle \cdot, \cdot \rangle_\lambda$, $\lambda \geqslant 1$. Let $H_0$ be a nontrivial closed subspace of $H$ such that $H_0 \neq H$. Assume the following conditions for the family of the scalar products. The induced norms are equivalent. If $\lambda = 1$, then it coincides with the original scalar product of $H$. The restrictions of the scalar products on $H_0 \times H$ are independent of $\lambda$ and coincide with the original scalar product. The family of scalar products cut the part contained in $H_0^\perp$, more precisely, for $\lambda_n \to \infty$, $(\|u_n\|_{\lambda_n})$ bounded and there is a subsequence $(\lambda_{n_k})$ and $u \in H_0$ such that $u_{n_k} \rightharpoonup u$ in $H$. Here $k : H \to \mathbb{R}$ is completely continuous, i.e.,

$$u_n \rightharpoonup u \text{ in } H \quad \Rightarrow \quad k(u_n) \to k(u).$$

The functional $J_0$ is defined by the restriction of $J_\lambda$ on $H_0$, which is independent of $\lambda$. A stable critical point $\bar{u}$ of $J_0$ is, for example, an isolated critical point with nontrivial critical groups $C_*(J_0, \bar{u})$. There are more results in the special cases if $\bar{u}$ is a local strict minimum or if $\bar{u}$ is a nondegenerated critical point. There are also more results in the general case.

# Spatio-Temporal Localization of Inner-Shell Excitations in Free Molecules, Clusters, and Solids

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Absorption of X-ray synchrotron and free-electron-laser radiation in matter is accompanied by strong dynamic core-hole localization and temporary trap of the electron ejected from a deep level within the finite-size potential barrier. As a result, the symmetry of core excited states is reduced as the inversion symmetry possessing in the ground state is broken. This is a very general property of core-excited polyatomic compounds with equivalent atoms: their equivalence implies their equal probability of excitation (the value averaged over large timescale), but not simultaneous core excitation. This means that one-photon absorption of the quasidegenerate core levels occurs in one of the equivalent atoms in a system, and the photoelectron wave function $\Psi$ should be presented as a symmetry-adapted linear combination of atomic functions $\phi_n$, which describe photoelectrons emitted from $n$-th equivalent position in a system. To take this "quasi-atomic-femtosecond" dynamic localization into account, different time dependence for the atomic wave functions is assumed in this superposition [1,2]:

$$\Psi(t) = \sum_n c_n \phi_n(t - t_n - \tau), \tag{1}$$

where $|c_n| = 1$, $t_n$ is the beginning of the $n$-core-ionization, and $\tau$ is the characteristic timescale of hole delocalization. Assuming that $|t_n - t_m| \gg T$, where $T$ is a time characterizing the interaction of photoelectrons with the anisotropic molecular (or cluster) potential, we infer that the photoelectron flux is a sum of incoherent "atomic" fluxes $J = \sum_n |\phi_n|^2$. In contrast, for $|t_n - t_m| \ll T$, we infer that $J = |\sum_n \phi_n|^2$. The variable phase approach is used to determine the functions $\phi_n$ (see, e.g., [2]).

The spatio-temporal (nanometric-femtosecond) dynamic localization of core-excitations in molecule, clusters, and solids is supported by examining the experimental data. In particular, the photoelectron angular distributions from $N$ and $O$ 1s levels in fixed-in-space $N_2$ and $CO_2$ molecules [1], the photoelectron induced rotational heating of $N_2$ [3], the Auger decay spectra of $N_2$ [4] and the near $F$ 1s edge X-ray absorption fine structure of free $SF_6$ molecules, free molecular $SF_6$ clusters and $SF_6$-solids. The results are discussed in more detail. It is shown that core-hole hoping and electronic relaxation tend to restore the ground symmetry. In contrast, relaxation of nuclear subsystem (in particular, excitation of low-symmetric molecular vibrations, photoelectron-induced recoil, and fragmentation) tends to retain the broken symmetry.

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# Bisectorial Operator Pencils and a Bounded-Solutions Problem

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Let $X$ and $Y$ be Banach spaces. Let us denote the space of all bounded linear operators acting from $X$ to $Y$ by $\mathcal{B}(X,Y)$.

Let $F,G\in\mathcal{B}(X,Y)$. The function $\lambda\mapsto\lambda F-G$, $\lambda\in\mathbb{C}$, is called an *operator pencil*. We call the pencil $\lambda\mapsto\lambda F-G$ *bisectorial* if there exist $\delta_0\in(0,\pi/2]$ and $h_0>0$ such that the resolvent set of the pencil contains the set

$$\Omega_{\delta_0,h_0}=\Big\{\lambda\in\mathbb{C}\colon|\arg\lambda|<\frac{\pi}{2}+\delta_0,\ \frac{\pi}{2}-\delta_0<\arg\lambda<-\frac{3\pi}{2}+\delta_0\Big\}\cup\{\lambda\in\mathbb{C}\colon|\operatorname{Re}\lambda|<h_0\},$$

and for all $\delta\in(0,\delta_0)$ and $h\in(0,h_0)$ there exist $M\in\mathbb{R}$ and $m\in\mathbb{Z}$ such that

$$\|(\lambda F-G)^{-1}:Y\to X\|\leqslant M(1+|\lambda|)^m,\qquad\lambda\in\Omega_{\delta,h}.$$

Assume that there exists a linear subspace $Y^1\subseteq Y$ such that it is complete in its norm $\|\cdot\|_1$ and $\|y\|\leqslant\|y\|_1$ for $y\in Y^1$.

We say that the pencil $\lambda\mapsto\lambda F-G$ is $Y^1$-*bisectorial* if there exist $\delta_0\in(0,\pi/2]$ and $h_0>0$ such that the resolvent set of the pencil contains the set $\Omega_{\delta_0,h_0}$ and for all $\delta\in(0,\delta_0)$ and $h\in(0,h_0)$ there exists $M\in\mathbb{R}$ such that

$$\|(\lambda F-G)^{-1}:Y^1\to X\|\leqslant\frac{M}{1+|\lambda|},\qquad\lambda\in\Omega_{\delta,h}.$$

Let us denote the set of all continuous bounded functions $f:\mathbb{R}\to Y$ by $C=C(\mathbb{R},Y)$. Similarly, we denote the set of all differentiable functions $u:\mathbb{R}\to X$ that are continuous and bounded with their derivative by $C^1=C^1(\mathbb{R},X)$.

We define the *Green function* by the formula

$$\mathcal{G}(t)=\begin{cases}\frac{1}{2\pi i}\int_{\Gamma^+_{\delta,h}}e^{\lambda t}(\lambda F-G)^{-1}\,d\lambda & \text{for t>0},\\ -\frac{1}{2\pi i}\int_{\Gamma^-_{\delta,h}}e^{\lambda t}(\lambda F-G)^{-1}\,d\lambda & \text{for t<0},\end{cases}$$

where $\Gamma^+_{\delta,h}$ ($\Gamma^-_{\delta,h}$) is the left (right) boundary of the set $\Omega_{\delta,h}$ for some $\delta\in(0,\delta_0)$, $h\in(0,h_0)$. The Green function exponentially decreases at infinity and has a summable singularity at the origin.

**Theorem.** *Let the pencil $\lambda\mapsto\lambda F-G$, $\lambda\in\mathbb{C}$, be $Y^1$-bisectorial. Then for any function $f\in C(\mathbb{R},Y^1)$ the equation*

$$\big(Fu\big)'(t)-Gu(t)=f(t),\qquad t\in\mathbb{R},$$

*has a unique solution $u \in C^1(\mathbb{R}, X)$. This solution can be represented in the form*

$$u(t) = \int_{-\infty}^{+\infty} \mathcal{G}(t-s)f(s)\,ds, \qquad t \in \mathbb{R}.$$

## Homogenization of Spin Energies


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The talk will focus on homogenization and $\Gamma$-convergence of surface and line energies defined on lattice (spin) systems in $\mathbf{Z}^d$ through bond interactions. We will dwell on nearest neighbours interaction systems and consider both periodic and random statistically homogeneous ergodic cases.

Given a smooth bounded domain $G \subset \mathbf{R}^n$ and a small parameter $\varepsilon > 0$, we denote $\varepsilon\mathbf{Z}^d \cup G$ by $G_\varepsilon$ and, for a function $u$ defined on $G_\varepsilon$, consider the energy

$$E_\varepsilon(u) = \sum_{i,j \in G_\varepsilon} \varepsilon^{d-1} c_{ij}(u_i - u_j)^2, \qquad c_{ij} \geqslant 0, \qquad c_{ij} = 0 \text{ if } |i-j| \neq \varepsilon.$$

Our goal is to study the limit behaviour of $E_\varepsilon$ as $\varepsilon \to 0$.

## Blow-Up Solutions to the Korteweg–de Vries Equation


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We deal with singular solutions to the Korteweg–de Vries equation which blow up at finite time. We examine solutions both for initial-boundary value problems and for the Cauchy problem.

We give an estimate of blow-up time depending on the boundary and initial conditions.

In conclusion, we demonstrate the explicit form of the blow-up solution of the Cauchy problem and show the mechanism of the blow-up of this solution.

# Smoothness of Generalized Solutions of Elliptic Functional-Differential Equations with Degeneration

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We consider elliptic functional-differential operators with degeneration of the second order. Unlike strongly elliptic functional-differential equations, smoothness of generalized solutions of elliptic functional-differential equations with degeneration can be violated [1,2]. Moreover, generally speaking, a generalized solution of such equation does not belong even to the Sobolev spaces of the first order. We obtain a priori estimates of solutions of elliptic functional-differential equation with degeneration [3]. Using these estimates, we prove that the orthogonal projection of a generalized solution to the image of the difference operator belongs to the Sobolev space of the second order in the subdomains (up to their boundaries) without the conjugation points.

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# Implicit Difference Methods for Nonlinear First-Order Partial Functional-Differential Equations

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We consider the nonlinear functional-differential equation

$$\partial_t z(t,x) = f(t,x,z(t,x),z_{\varphi(t,x)},\partial_x z(t,x)) \tag{1}$$

with initial condition

$$z(t,x) = \varphi(t,x) \quad \text{for} \quad (t,x) \in [-b_0,0] \times [-b,b], \tag{2}$$

where $x = (x_1,\dots,x_n)$, $\partial_x z = (\partial_{x_1} z,\dots,\partial_{x_n} z)$, $b_0 \in \mathbb{R}_+$, $b = (b_1,\dots,b_n)$, and $b_i > 0$ for $1 \leqslant i \leqslant n$. The functional argument is presented by $z_{\varphi(t,x)}$.

Classical solutions are approximated by solutions of suitable quasilinear systems of functional-difference equations. The numerical methods are difference schemes implicit with respect to time variable. Theorems on the convergence of difference schemes and error estimates of approximate solutions are presented. The proof of the stability is based on a comparison technique with nonlinear estimates of the Perron type. Numerical examples are given.

The differential equations with deviated variable and integrodifferential equations are the particular cases of (1).

The lecture is based on the article [1].

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## The Oriented Degree for Compact Perturbations of Fredholm Nonlinear Maps and Bifurcation Theorem for Elliptic Boundary Value Problem


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Let $f+k$ be a map from a Banach manifold $X$ to a Banach space $E$, where $f$ is a proper nonlinear Fredholm map with zero index and $k$ is a compact continuous map. The map $f$ generates the Fredholm structure on $X$, i. e., the collection of charts $\{(U_i,\phi_i)\}$ with property $D(f\circ\phi_i^{-1})(x)\in GL_c(E)$. Assuming that the Fredholm structure is orientable and $X^+$ is an oriented subatlas, we present the construction of an oriented degree $d(f+k,X^+,y)$, $y\in E\backslash(f+k)(\partial X)$, based on the finite-dimensional reduction method.

We use the degree to examine the local and global bifurcation for the following elliptic boundary-value problem with real-valued parameter $\lambda$:

$$F(x,u,\dots,D^{2m}u)-\lambda u=G(x,u,\dots,D^{2m-1}u,\lambda),\quad x\in\Omega, \tag{1}$$

$$B_j(x,D)u=0,\quad j=0,\dots,m-1,\quad x\in\partial\Omega. \tag{2}$$

The above boundary-value problem generates the operator equation $f(u)-\lambda u-g(u,\lambda)=0$ in Sobolev spaces with a proper Fredholm map $f$ and continuous compact map $g$. We consider the supplementary equation

$$\Phi_r(u,\lambda)=(f(u)-\lambda u-g(u,\lambda),h_r(u))=0, \tag{3}$$

where $h_r=\|u\|^2_{W^{2m+1,p}(\Omega)}-r^2$. A solution of Eq. (3) is a nontrivial solution to the boundary-value problem (1)-(2) with $\|u\|_{W^{2m+1,p}(\Omega)}=r$.

**Theorem 1.** *Let $\lambda_0$ be an eigenvalue of $L_0=Df(0)$ with spectral multiplicity $m$. Then*

$$d(\Phi_r,D_\Phi^+,0)=\begin{cases} 0, & if\quad m\quad even,\\ \pm 2, & if\quad m\quad odd.\end{cases}$$

*There are two obstructions to detect the sign + or − in the above formula. Changes of the sign may appear during homotopy of a Fredholm nonlinear map, and there is no canonical choice of orientation on the neighbourhood of the point $(0,\lambda_0)\in W^{2m+1,p}(\Omega)\times R$.*

**Corollary.** *If $\lambda_0$ is an eigenvalue of operator $L_0$ with odd multiplicity, then $\lambda_0$ is a point of bifurcation to problem* (1)-(2).

**Theorem 2.** *Let $W$ be the closure of the set of all nontrivial solution to the problem* (1)-(2)*, $\lambda_0$ be a bifurcation point, and $W_0$ be the connected component of $W$ that contains $(0, \lambda_0)$. Then $W_0$ is unbounded or, if $W_0$ is bounded, then it contains finite set of bifurcation points $(0, \lambda_i)$. The number of points $(0, \lambda_i) \in W_0$ such that the eigenvalue $\lambda_i$ has odd multiplicity is even.*

# An Approach to the Description of Rotating Waves in Parabolic Functional-Differential Equations with Rotation of Spatial Arguments and Time Delay

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We consider the functional-differential parabolic equation with delay and rotation of spatial argument under periodic boundary condition:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - u + K(1 + \gamma \cos u(x + \theta, t - T)),\ u(0, t) = u(2\pi, t). \qquad (1)$$

This problem arises in the modelling of nonlinear optical systems with nonlocal delayed feedback loop in the case of a thin ring aperture [1]. Here $D > 0$, $T > 0$, $K > 0$, $\gamma \in (0, 1)$, and $\theta \in [0, 2\pi)$. Our aim is to study rotating waves branching off the homogeneous steady state as a result of the Andronov–Hopf bifurcation.

There are several approaches to dealing with the Andronov–Hopf bifurcation in case of delayed partial differential equations. For instance, in [2], the authors use the theory of semigroups combined with the implicit function theorem, while the method of normal forms is applied in [3] to reduce the problem to an ordinary differential system in the vicinity of the steady state. These methods are quite complicated in applications. At the same time, we can simplify a technique, using a symmetry property of a ring in Eq. (1).

In the present paper, we propose to apply a transition to rotating coordinates. This idea is well known for the case of differential equations without delay as a powerful tool to study travelling waves [4] and rotating waves [5]. Here we adopt the method to delayed parabolic equation. As a result, we obtain a 1-D boundary-value problem for stationary differential equation with deviated argument. This problem governs the shape of rotating waves. Using the implicit function theorem, we have proven the existence and uniqueness of rotating waves in a ring under usual Andronov–Hopf bifurcation conditions and obtained the coefficients of the corresponding small parameter expansion. The same results have been obtained for a 2-D statement in a circle.

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# Stability Analysis for Maxwell's Equation with a Thermal Effect for One Spatial Dimension

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We consider a microwave heating problem represented by a system, where the microwave radiation is described by Maxwell's equations and the heat transfer in some material is represented by a diffusion equation.

For one spatial dimension, this problem is represented (as formulated in [1, 2]) by the system

$$\begin{array}{ll} w_{tt} = w_{xx} - \sigma(\theta) w_t, & x \in (0,1),\ t > 0, \\ \theta_t = \theta_{xx} + \sigma(\theta) w_t^2, & x \in (0,1),\ t > 0, \\ w(0,t) = 0,\ w(1,t) = 0, & t > 0, \\ \theta(0,t) = \theta(1,t) = 0, & t > 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), & x \in (0,1), \\ \theta(x,0) = \theta_0(x), & x \in (0,1), \end{array} \tag{1}$$

where $w(x,t)$ is the solution component of the modified Maxwell's equation, $\theta(x,t)$ is the temperature, $\sigma(\theta)$ is the nonlinear electrical conductivity, and $w_0(x)$, $w_1(x)$, and $\theta_0(x)$ are some given functions. Our main problem is the investigation of the asymptotic behavior of the solutions of system (1). Under certain conditions, the exponential convergence of solution components $w$, $w_t$, and $\theta$ to zero was shown in [1, 3].

Further, we extend system (1) according to the heat equation in order to describe the phase-change process. This leads to a modified heat equation with a monotone enthalpy operator. We show that under certain conditions, all solutions of the system describing the phase-change process converge to stationary ones.

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# Simplified Approach to a Uniqueness Problem of a Nonautonomous Planar System

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We present a dynamical systems approach for handling uniqueness, problems in nonautonomous planar systems. The method is shown, while being applied to a uniqueness problem for a ground state of the continuum limit of a strictly nonharmonic multidimensional lattice. Although this problem, and even a more general version of it, was already solved by Pucci and Serrin, we believe that our method sheds a new light on it.

Our solution to the uniqueness problem has two parts. In the first one we show, using classical phase plane analysis tools, that solutions in the first quadrant of the plane cannot cross each other even though the system is nonautonomous. The second part presents a local uniqueness result, i.e., the one that is independent of the initial condition, when analyzing what happens near the origin. In addition, we demonstrate the usage of our phase plane analysis tools by proving a simple non-intersection result for some nonautonomous system in the plane.

# Functional-Differential Equations with Rescaling: the Gårding-Type Inequality

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Let $q > 1$ and $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ such that $\overline{\Omega} \subset q\Omega$. We consider the following functional-differential operator in $\Omega$:

$$Au(x) = \sum_{|\alpha|,|\beta|\leqslant m} D^\alpha \left[a_{\alpha\beta 0}(x)D^\beta u(x) + a_{\alpha\beta 1}(x)D^\beta u(q^{-1}x) + a_{\alpha\beta,-1}(x)D^\beta u(qx)\right], \tag{1}$$

where the coefficients $a_{\alpha\beta j}(x)$ are smooth functions in $\overline{\Omega}$, and establish some necessary conditions and sufficient conditions for the Gårding-type inequality

$$\operatorname{Re}(Au, u)_{L_2(\Omega)} \geqslant c_1\|u\|^2_{H^m(\Omega)} - c_2\|u\|^2_{L_2(\Omega)} \quad (u \in C_0^\infty(\Omega)). \tag{2}$$

If $A$ is a differential operator (e.g., $a_{\alpha\beta 1} = a_{\alpha\beta,-1} = 0$ in (1)), then (2) is a synonym of the strong ellipticity [1]. For a broader class of operators, inequality (2) guarantees the Fredholm solvability as well as discreteness and sectorial structure of the spectrum of the Dirichlét problem for the equation $Au = f$ in $L_2(\Omega)$. The fulfilment of (2) in the case of differential-difference equations was studied in [2].

**Theorem 1.** *Let inequality* (2) *hold for the operator $A$ given by* (1). *Then the self-adjoint part of the operator*

$$v(x) \mapsto \sum_{|\alpha|,|\beta|=m} \xi^{\alpha+\beta}\left[a_{\alpha\beta 0}(x)v(x) + a_{\alpha\beta 1}(x)v(q^{-1}x) + a_{\alpha\beta,-1}(x)v(qx)\right]$$

*is positive definite in $L_2(\Omega)$ for all $\xi \in S^{n-1}$.*

Introduce the notation $a_{\alpha\beta}(x) = \operatorname{Re} a_{\alpha\beta 0}(x)$, $b_{\alpha\beta}(x) = (a_{\alpha\beta 1}(x) + q^{-n}\bar{a}_{\alpha\beta,-1}(q^{-1}x))/2$,

$$a(x,\xi) = \sum a_{\alpha\beta}(x)\xi^{\alpha+\beta}, \quad b(x,\xi) = \sum b_{\alpha\beta}(x)\xi^{\alpha+\beta}, \quad r(x,\xi) = \frac{q^{n/2}b(x,\xi)}{\sqrt{a(x,\xi)a(q^{-1}x,\xi)}}$$

(the summation is over all $|\alpha|, |\beta| = m$). It is a simple consequence of Theorem 1 that $a(x,\xi)$ is positive in $\overline{\Omega}$.

**Theorem 2.** *If there exists a smooth function $\delta(x,\xi)$ such that $0 < \delta(x,\xi) < 1$ and*

$$|r(x,\xi)|^2 < \delta(q^{-1}x,\xi)\,(1-\delta(x,\xi)) \quad (x \in \overline{\Omega}, \xi \in S^{n-1}),$$

*then inequality* (2) *holds for the operator $A$ given by* (1).

**Example 1.** If $r(x,\xi) < 1/4$ for $x \in \overline{\Omega}, \xi \in S^{n-1}$, then $A$ satisfies (2).

It should be noted that the condition $r(\xi) < 1/4$ coincides with the necessary condition from Theorem 1 in the case where the coefficients $a_{\alpha\beta j}(x)$ are constants [3].

**Example 2.** Take $\delta(x,\xi) = ke^{-q^2|x|^2}$ with $(q^2+1)^{-1} < k < 1$ and $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$ with $R > q^{-1}\sqrt{\ln k(q^2+1)}$. Then we get the sufficient condition

$$|r(x,\xi)|^2 < ke^{|x|^2}\left(1 - ke^{-q^2|x|^2}\right) \quad (|x| < R)\,.$$

This example shows that $|r(x,\xi)|$ is allowed to be arbitrarily close to 1 at some points of the domain.

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# Analysis and Control of Photon-Induced Processes

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Photon induced processes occur on different time scales, which may range from milliseconds to the attosecond time regime. Ultrashort pulses of lasers are suitable to study ultrafast photon induced processes in the time domain. This reveals the dynamics of excited states, electron motion, electron emission, and molecular fragmentation. A fundamental work on atomic, molecular, and nanoscopic systems is presented in this context. In addition to the pure description of ultrafast processes by state-of-the-art experimental approaches, it is possible to control them. This is accomplished by suitable control schemes, which include coherent control by using laser sources, pulse shaping and chirping, as well as phase control of laser pulses. This allows one to optimize photon induced processes, as evidenced by various examples from recent researches.

## Equiconvergence Theorems for Sturm–Liouville Operators with Singular Potentials

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We deal with the Sturm–Liouville operator

$$L = -\frac{d^2}{dx^2} + q(x)$$

with Dirichlet boundary conditions $y(0) = y(\pi) = 0$ in the space $L_2[0,\pi]$. We assume that the potential $q$ is complex-valued and has the form $q(x) = u'(x)$, where $u \in L_2[0,\pi]$. Here the derivative is treated in the distributional sense. This class of operators was defined in the paper of A. M. Savchuk and A. A. Shkalikov [1]. We consider the problem of equiconvergence (in $C[0,\pi]$-norm) of two expansion of a function $f \in L_1[0,\pi]$. The first one is constructed using the system of the eigenfunctions and associated functions of the operator $L$, while the second one is the Fourier expansion in the series of sines.

**Theorem 1.** *Consider operator $L$ acting in the space $L_2[0,\pi]$ with the Dirichlet boundary conditions. Suppose that the complex-valued potential $q(x)$ is equal to $u'(x)$, where $u \in L_\infty[0,\pi]$ and*

$$\sup_{0\leqslant x\leqslant\pi}\ \sup_{0<h\leqslant\pi} \int\limits_{h\leqslant|t|\leqslant\pi} \left|\frac{u(t+x+h)-u(t+x)}{t}\right| dt \leqslant C < +\infty$$

(we assume that $u$ is a $\pi-$periodic function). Let $\{y_n(x)\}_{n=1}^\infty$ be the system of the eigenfunctions and associated functions of the operator $L$ and $\{w_n(x)\}_{n=1}^\infty$ be the biorthogonal system.

For an arbitrary function $f \in L_1[0,\pi]$, denote

$$c_n := (f(x), w_n(x)), \quad c_{n,0} := \sqrt{2/\pi}(f(x), \sin nx).$$

Then

$$\lim_{m\to\infty}\left\|\sum_{n=1}^m c_n y_n(x) - \sqrt{\frac{2}{\pi}}\sum_{n=1}^m c_{n,0}\sin(nx)\right\|_{C[0,\pi]} = 0.$$

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# On Variational Description of the Trajectories of Averaging Semigroups

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We consider the Cauchy problem for the Schrödinger equation with operator $\mathbf{L}$ in the Hilbert space $H$ given by the second-order differential expression $\mathbf{L}u = \frac{\partial}{\partial x}(g(x)\frac{\partial}{\partial x}u) + \frac{i}{2}[a(x)\frac{\partial}{\partial x}u + \frac{\partial}{\partial x}(a(x)u)]$. Here $g(x)$ and $a(x)$ are real-valued step function and $g(x)$ is nonnegative. The Cauchy problem is ill-posed due to the vanishing of the function $g(x)$ on some set. In particular, the interval of the existence of the solution depends on the initial data and can be finite.

The regularization of the degenerated operator $\mathbf{L}$ is the sequence $L_\varepsilon$, $\varepsilon \in E = (0,1)$, $\varepsilon \to 0$, of regularized self-adjoint operators with the coefficients $g_\varepsilon(x) = g(x)+\varepsilon$ instead of $g(x)$. We investigate the sequence $T_\varepsilon(t)$, $t > 0$, $\varepsilon \in E$, $\varepsilon \to 0$, of regularizing dynamical semigroups of transformations of the Banach space $X = B^*(H)$ of linear continuous functionals on the Banach space $X_* = B(H)$ of bounded linear operators in the space $H$ acting by the rules $(T_\varepsilon(t)\rho, \mathbf{A}) = (\rho, e^{-i\mathbf{L}_\varepsilon t}\mathbf{A}(e^{-i\mathbf{L}_\varepsilon t})^*)$, $(t, \rho, \mathbf{A}) \in R \times X \times X_*$.

**Theorem 1.** *Let the deficiency indices $(n_-, n_+)$ of the operator $\mathbf{L}$ be finite and the set of operators $\mathbf{L}_\theta$, $\theta \in \Theta$, be the collection of maximal symmetric dilatations of the operator $\mathbf{L}$. Then the set of values of the sequence $\{T_\varepsilon(t)\}$ is sequentially compact in the weak-* operator topology of the space $B(X)$ if and only if $n_+ \leqslant n_-$. In this case, the set of particular limits of the sequence $\{T_\varepsilon(t)\}$ in the above topology belongs to the set of semigroups $T^\theta(t)$, $\theta \in \Theta$, acting by the rules $(T^\theta(t)\rho, \mathbf{A}) = (\rho, e^{-i\mathbf{L}_\theta t}\mathbf{A}(e^{-i\mathbf{L}_\theta t})^*)$, $(t, \rho, \mathbf{A}) \in R_+ \times X \times X_*$. In the other case $n_+ > n_-$, the set of particular limits of the sequence $\{T_\varepsilon(t)\}$ is empty.*

Let $W(E)$ be the set of nonnegative normalized bounded additive measures on the algebra of all subsets $2^E$ of the set $E$ of regularization parameters, which is concentrated in an arbitrary neighborhood of the limit point $\varepsilon_0 = 0$ (see [1]). The family of averaging dynamical maps $T^\mu(t)$, $t > 0$, is defined as the Pettis integral of the operator-function $T_\varepsilon(t)$, $\varepsilon \in E$, by the measure $\mu$ in the weak-* operator topology, i.e., the equality $(T^\mu(t)\rho, \mathbf{A}) = \int\limits_E (T_{\varepsilon_k}(t)\rho, \mathbf{A})d\mu$ holds for any $\rho \in X$, $\mathbf{A} \in X_*$, and $t > 0$.

**Theorem 2.** *Let $\mathbf{L}$ be a symmetric operator in the space $H$ and $\mathbf{L}_\varepsilon$, $\varepsilon \in E$, be its regularization. Then the set of limit points of the sequence $T_\varepsilon(t)\rho$ in the weak-* topology coincides with the set $\bigcup\limits_{\mu \in W(E)} T^\mu(t)\rho$ for any $(t, \rho) \in R_+ \times X$.*

The family of averaging maps $T^\mu(t)$, $t > 0$, does not possess the semigroup property and for any $t > 0$ the map $T^\mu(t)$ is not injective. However, the trajectory of the averaging maps $T^\mu(t)\rho_{u_0}$, $t > 0$, can be found as the solution of some variational problem. Let $S_1^+(X)$ be the set of nonnegative continuous linear functionals of the space $X_*$ with the unit norm and $\Sigma(H)$ be the set $\{\rho \in S_1^+(X) : \exists u \in H, \|u\|_H = 1, \langle \rho, \mathbf{A}\rangle = (u, \mathbf{A}u) \equiv \langle \rho_u, \mathbf{A}\rangle\}$.

**Theorem 3.** *Let the conditions of theorem 2 hold. Then there exists a set $M \subset W(E)$ such that for any $\mu \in M$ the family of maps $T^{\mu}(t)$, $t > 0$, possesses the following property*:

There exists a function $F_{\mu}: \ R_{+} \times S_1^{+}(X) \times \Sigma(H) \ \to \ R$ such that for any $u_0 \in H$ and $t > 0$ the set $C(t) = \mathrm{argmax}(F_{\mu}(t, T^{\mu}(t)\rho_{u_0}, \cdot))$ is diffeomorphic to the circle, and there exists $t_1 > 0$ such that $\rho_{u_0} = C(t) \bigcap C(t_1)$.

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# Navier–Stokes Equations: on the Problem of Turbulence


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The Cauchy problem for the 3D Navier–Stokes equations with periodical condition on the spatial variable is researched. The vector functions under consideration are decomposed in Fourier series with respect to eigenfunctions of the curl (rotor) operator. The problem is reduced to a Cauchy problem for the Galerkin systems of of differential equations, which has a simplest structure in the considered basis. The following programs are made up: reconstruction for the Galerkin systems and numerical solution of the Cauchy problem. Several prototype problems are solved. The results are represented in the graphic form, which illustrates the turbulent flows of the liquid and the appearance of whirls. The Cauchy problem for the linear Stokes equations is investigated. The families of Gilbert spaces are chosen. I prove that the operator of the problem realizes isomorphism of the corresponding spaces. In some cases the exact solutions of nonlinear Navier–Stokes equations are found. Moreover, two Gilbert spaces are written out and I prove that the sequence of Galerkin approximations is limited at each of these spaces.


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# Spectral Properties of Dirac Operators on $(0,1)$ with Summable Potentials

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We consider the Dirac operator $L$ generated in the space $(L_2[0,1])^2$ by the differential expression

$$B\frac{d}{dx}+Q, \qquad B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad Q=\begin{pmatrix}q_1 & q_2\\ q_3 & q_4\end{pmatrix},$$

and some regular boundary conditions. We assume that $Q$ belongs to $L_p[0,1]$ for some $p\in[1,\infty)$ or to the Sobolev space $W_2^\theta[0,1]$ with some $\theta\in[0,1/2)$. For such kind of potentials, we establish an asymptotic behavior of eigenvalues and eigenfunctions of the operator $L$. In the general situation (when the entries of $Q$ belong to $L_1(0,1)$), we prove that the system of eigenfunctions and associated functions form a Riesz basis in $(L_2[0,1])^2$.

The talk is based on joint works with A. A. Shkalikov.

# Uniformization Problem in Nonlocal Elliptic Theory

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Let $M$ be a compact smooth manifold and $G$ be a compact Lie group acting on $M$ by diffeomorphisms. On $M$, we consider nonlocal operators of the form

$$D=\int_G D_g T_g dg : C^\infty(M)\longrightarrow C^\infty(M), \tag{1}$$

where $D_g$, $g\in G$, is a smooth family of pseudodifferential operators ($\psi$DO), $T_g$ is the representation of $G$ by shift operators on the space of functions on $M$, and $dg$ is the Haar measure on $G$.

The operators of the form (1) will be called *G-pseudodifferential operators*. Note that, if $G$ is trivial, then the set of $G$-pseudodifferential operators coincides with the set of classical pseudodifferential operators.

If $G$ is a discrete group, then the integral in (1) reduces to a sum over $G$, and the corresponding $G$-pseudodifferential operators were studied by many authors (e.g., Connes [1], Antonevich, and Lebedev [2] and others). The index formula for such operators was obtained in [3].

Here we construct the elliptic theory for operators of the form (1) for a general compact Lie group $G$. To this end, we represent our $G$-pseudodifferential operators as classical pseudodifferential operators acting in sections of infinite-dimensional bundles, whose fiber is the space of functions on $G$. This method (which we call *pseudodifferential uniformization*) goes back to Ch. Babbage, and for finite groups gives a finite system of equations. In addition, the obtained operator, which we denote by $\mathcal{D}$, is $G$-invariant, and its restriction $\mathcal{D}^G$ to the space of $G$-invariant sections turns out to be isomorphic to the original operator $D$. Now, if an operator $\widehat{\mathcal{D}} = 1 + \mathcal{D}$ satisfies the transversal ellipticity condition for $G$ (this condition was introduced by Atiyah and Singer [4] for operators acting in sections of finite-dimensional bundles), then this implies the finiteness theorem (Fredholm property), i.e., the index of the operator $\widehat{D} = 1 + D$ is finite.

We also apply pseudodifferential uniformization to obtain an index formula.

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# Control of Delayed Complex Networks


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Time delays arise naturally in many complex networks, for instance, in neural networks, as delayed coupling or delayed feedback due to finite signal transmission and processing times [1].

Such time delays can either induce instabilities, multistability, and complex bifurcations, or suppress instabilities and stabilize unstable states. Thus, they can be used to control the dynamics [2].

We study the synchronization in delay-coupled oscillator networks, using a master stability function approach [3]. Within a generic model of the Stuart–Landau oscillators (a normal form of supercritical Hopf bifurcation) we derive analytical stability conditions and demonstrate that one can easily control the stability of synchronous periodic states by tuning the coupling phase.

We propose the coupling phase as a crucial control parameter to switch between in-phase synchronization or desynchronization for general network topologies or between in-phase, cluster, or splay states in unidirectional rings. Our results are robust even for slightly nonidentical elements of the network. We also discuss applications to neural networks, in particular, small-world networks with inhibitory couplings, and to chaotic laser networks.

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# Adiabatic Limit in Ginzburg–Landau Equations

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Hyperbolic Ginzburg–Landau equations are the Euler–Lagrange equations for the (2+1)-dimensional Abelian Higgs model, arising in gauge field theory. Static solutions of these equations are called vortices and their moduli space is described by Taubes. The structure of the moduli space of dynamic solutions is far from being understood, but there is an heuristic method, due to Manton, allowing to construct solutions of Ginzburg–Landau equations with small kinetic energy. The idea is that in the adiabatic limit dynamic solutions should converge to geodesics on the moduli space of vortices in the metric generated by kinetic energy functional. According to Manton's adiabatic principle, any solution of dynamic equations with a sufficiently small kinetic energy can be obtained as a perturbation of some geodesic of this type. Our talk is devoted to the mathematical justification of this principle.

# The Description of Freak Waves by Functional-Differential Inclusions

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In the present paper, we consider a dynamical system based on the functional-differential inclusions describing ideal fluid flow with a free surface in the presence of external influence. This dynamical system is applied to the analysis of large surface waves, i.e., freak waves.

As freak waves is not a usual effect, there is an actual question about their stability.

In the paper, it was ascertained that the solutions describing freak waves are stable in the sense of the initial data and external influence. Moreover, we carried out the large scale computing experiments demonstrating that freak waves are quite stable effect.

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# Homogenization of Boundary-Value Problems for the Laplace Operator in Perforated Domains with Nonlinear Third-Type Boundary Conditions on the Boundary of Cavities

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We consider the asymptotic behavior of the solutions $u_\varepsilon$ of the Poisson equation in $\varepsilon$-periodically perforated domain with a nonlinear third-type boundary condition on the perforated part of the boundary. We suppose that the diameter of the sets, which construct the perforation, is equivalent to $\varepsilon^\alpha$, where $\alpha > 1$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geqslant 3$, with a smooth boundary $\partial\Omega$, $G_0$ be the unit ball centered at the origin, and $Y = (-1/2, 1/2)^n$. For $\delta > 0$ and $\varepsilon > 0$, we denote $\{x \,|\, \delta^{-1}x \in B\}$ by $\delta B$ and $\{x \in \Omega \,|\, \rho(x, \partial\Omega) > 2\varepsilon\}$ by $\widetilde{\Omega}_\varepsilon$. We set $G_\varepsilon = \bigcup_{z\in\Upsilon_\varepsilon}(a_\varepsilon G_0 + \varepsilon z) = \bigcup_{j=1}^{N(\varepsilon)} G_\varepsilon^j$, where $a_\varepsilon = C_0\varepsilon^\alpha$, $C_0 > 0$, $\alpha > 1$, $\Upsilon_\varepsilon = \{z \in \mathbb{Z} : (a_\varepsilon G_0 + \varepsilon z) \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \varnothing\}$, $N(\varepsilon) \cong d\varepsilon^{-n}$, $d = const > 0$, and $\mathbb{Z}$ is the set of vectors with integer coordinates. We denote

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},\ S_\varepsilon = \partial G_\varepsilon,\ \partial\Omega_\varepsilon = \partial\Omega \bigcup S_\varepsilon.$$

In the domain $\Omega_\varepsilon$, we consider the boundary-value problem

$$-\Delta u_\varepsilon = f, x \in \Omega_\varepsilon, \quad \partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\sigma(x, u_\varepsilon) = 0, x \in S_\varepsilon, \quad u_\varepsilon = 0, x \in \partial\Omega, \tag{1}$$

where $f \in L_2(\Omega)$, $\nu$ is the exterior unit normal vector to $S_\varepsilon$, $\gamma \in \mathbb{R}$, $\sigma(x, u)$ is a continuously differentiable function of variables $x \in \overline{\Omega}$ and $u \in \mathbb{R}^1$, $\sigma(x, 0) = 0$, and there exist two positive constants $k_1$ and $k_2$ such that $k_1 \leqslant \partial_u\sigma(x, u) \leqslant k_2$.

**1.** Suppose that $\alpha \in [1, \frac{n}{n-2})$ and $\gamma > \alpha(n-1) - n$.
The following theorem is proved:

**Theorem 1.** *Suppose that $u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ is a solution of problem* (1)*, $\alpha \in [1, \frac{n}{n-2})$, and $\gamma > \alpha(n-1) - n$. Then $\|u_\varepsilon\|_{L_2(\Omega_\varepsilon)} \to 0$ as $\varepsilon \to 0$.*

**2.** Suppose that $\alpha = \frac{n}{n-2}$, $\gamma > \alpha(n-1) - n$, and $u_0 \in H^1(\Omega)$ is a weak solution of the boundary - value problem

$$-\Delta u_0 + C_1 u_0 = f \text{ in } \Omega, u_0(x) = 0 \text{ on } \partial\Omega, \tag{2}$$

where $C_1 = (n-2)C^{n-2}\omega_n$, $\omega_n$ is the area of the unit sphere in $R^n$.

**Theorem 2.** *Suppose that $u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ is a solution of problem* (1)*, $u_0$ is a solution of problem* (2)*, $\alpha = \frac{n}{n-2}$, and $\gamma > \alpha(n-1) - n$. Then $u_\varepsilon \rightharpoonup u_0$ in $H_0^1(\Omega_\varepsilon)$ as $\varepsilon \to 0$.*

# Painting Chaos and Global Bifurcations: Universality of the Lorenz Attractor

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We show that "painting" the kneading invariants for the separatrices of a saddle allows for uncovering hidden structures, such as codimension-two T-points that organize globally the parameter space and dynamics of a system (examples will be presented) with the Lorenz attractor. This is a joint work with R. Barrio (University of Zaragoza, Spain).

# Pseudo-Hyperbolic Attractors

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Usually, by a chaoticity of a given system in some bounded domain, one understands such behavior of its orbits that neither the system itself nor any system close to it have a stable periodic motion. A formalization of this phenomenon in the spirit of the Anosov's hyperbolicity usually bears the name the "pseudo-hyperbolicity". Here, by a pseudo-hyperbolic attractor, a stable chain-transitive invariant set is understood. Such an attractor always exists in an absorbing domain in accordance to Conley–Ruelle. Here we consider the cases when the singular element of the attractor is its periodic orbit. A simplest example here is a Lorenz attractor or spiral attractor under the influence of a small periodic force. A feature of these attractors is their wildness, that is the presence, as a rule, of homoclinic tangencies. Another example is the attractor emerging as a result of the Andronov–Hopf bifurcation leading to the emergence from a saddle-focus a saddle periodic orbit. To this type of pseudo-hyperbolic attractors, a problem of bifurcations in 4-dimensional systems with homoclinic tangencies leads. In the case where a saddle periodic orbit has two-dimensional unstable manifold, a sufficiently general existence theory for pseudo-hyperbolic attractors will be constructed with the indication of those cases where the transitivity on the attractor takes place.

# Fading Absorption in Semilinear Elliptic Equations

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We study the Dirichlet problem in the model domain $\Omega = \mathbb{R}^N_+ = \{(x', x_N) : x_N > 0\}$ for the equation

$$-\Delta u + h(x')|u|^{q-1}u = 0, \qquad q > 1,$$

with singular data at the boundary $\Gamma = \{x \in \mathbb{R}^N : x_N = 0\}$. Here $h(x')$ is a smooth degenerate potential: $h(x') > 0 \ \forall x' \neq 0$, and $h(0) = 0$. In the case of $h(|x'|)$, we find a sufficient and necessary condition (criterium) for the flatness of $h(s)$ near to the point $s = 0$, guaranteeing the existence (nonexistence) of a "large" solution (solution with boundary data $u = \infty$ at $\Gamma$), of a very singular solution (nonnegative solution with $u(x', 0) = 0 \quad \forall x' \neq 0$, and "strong" point singularity at $x = 0$).

Joint results with Moshe Marcus.

## Old and New in Complex Dynamical Systems

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Historically, complex dynamics and geometrical function theory have been intensively developed from the beginning of the twentieth century. They provide the foundations for broad areas of mathematics. In the last fifty years, the theory of holomorphic mappings on complex spaces has been studied by many mathematicians with many applications to nonlinear analysis, functional analysis, differential equations, classical and quantum mechanics. The laws of dynamics are usually presented as equations of motion, which are written in the abstract form of a dynamical system:

$$((dx)/(dt)) + f(x) = 0,$$

where $x$ is a variable describing the state of the system under study and $f$ is a vector-function of $x$. The study of such systems when $f$ is a monotone or an accretive (generally nonlinear) operator on the underlying space has recently been the subject of much research by analysts working on quite a variety of interesting topics, including boundary value problems, integral equations, and evolution problems.

In this talk, we give a brief description of the classical statements, which combine the celebrated Julia Theorem of 1920, Carathéodory's contribution in 1929, and Wolff's boundary version of the Schwarz Lemma of 1926 with their modern interpretations for discrete and continuous semigroups of hyperbolically nonexpansive mappings in Hilbert spaces. We also present flow-invariance conditions for holomorphic and hyperbolically monotone mappings.

Finally, we study the asymptotic behavior of one-parameter continuous semigroups (flows) of holomorphic mappings. We present angular characteristics of the flows trajectories at their Denjoy–Wolff points as well as at their regular repelling points (whenever they exist). This enables us by using linearization models in the spirit of functional Schröder's and Abel's equations and eigenvalue problems for composition operators to establish new rigidity properties of holomorphic generators, which cover the famous Burns–Krantz theorem, and to solve a Nevanlinna–Pick-type boundary interpolation problem for generators.

# Bifurcations of Solutions of PDEs

Ya. G. Sinai
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We discuss several cases of bifurcations produced by solutions of linear and nonlinear PDEs.

# Classical Solutions of Boundary-Value Problems for the Vlasov Equations in a Half-Space

A. L. Skubachevskii
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We consider the Vlasov system of equations describing the evolution of distribution functions of the density for the charged particles in a rarefied plasma. We study the Vlasov system in $\mathbb{R}^3_+ \times \mathbb{R}^3$ with initial conditions for distribution functions $f^\beta\big|_{t=0} = f_0^\beta(x,p)$, $\beta = \pm 1$, and the Dirichlet or Neumann boundary conditions for the potential of an electric field for $x_1 = 0$, where $f_0^\beta(x,p)$ is the initial distribution function (for positively charged ions if $\beta = +1$ and for electrons if $\beta = -1$) at the point $x$ with impulse $p$, $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \colon x_1 > 0\}$. Assume that initial distribution functions are sufficiently smooth and $\operatorname{supp} f_0^\beta \subset (\mathbb{R}^3_\delta \cap B_\lambda(0)) \times B_\rho(0)$, $\delta, \lambda, \rho > 0$, and the magnetic field $H(x)$ is also sufficiently smooth and has a special structure near the boundary $x_1 = 0$, where $\mathbb{R}^3_\delta = \{x \in \mathbb{R}^3 \colon x_1 > \delta\}$. Then we prove that for any $T > 0$ there is a unique classical solution of the Vlasov system in $\mathbb{R}^3_+ \times \mathbb{R}^3$ for $0 < t < T$ if $\|f_0^\beta\| < \varepsilon$, where $\varepsilon = \varepsilon(T, \delta, \rho, \|H\|)$ is sufficiently small.

This work was supported by the RFBR (grants No. 10-01-00395 and No. 09-01-00586) and the analytical departmental special-purpose program "Development of Scientific Potential of Higher Education" (No. 2.1.1/5328).

# On Asymptotic of Solutions of Elliptic Boundary-Value Problems at Angular Points

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An elliptic system with constant principal coefficients is considered in the angle domain on the plane. The solution of this equation satisfies boundary conditions (nonlocal in general) on the lateral sides of the angle. The following question is discussed. Suppose that the right-hand parts of these boundary conditions have power-logarithmic asymptotic. Then, under what angular conditions this property is valid for the solution?

# On the Solvability of the Homogeneous Dirichlet Problem with the $p$-Laplacian Perturbed by a Difference Operator

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We consider the essentially nonlinear Dirichlet problem

$$\Delta_p R_Q u(x) = f_0(x) \qquad (x \in Q), \tag{1}$$

$$u(x) = 0 \qquad (x \in \partial Q). \tag{2}$$

Here $Q \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial Q$, $p \in (2, \infty)$, $1/p + 1/q = 1$, $f_0 \in L_q(Q)$, and $\Delta_p$ is the $p$-Laplacian given by the formula

$$\Delta_p u(x) = - \sum_{1 \leqslant i \leqslant n} \partial_i \left( |\partial_i u|^{p-2} \partial_i u \right).$$

A bounded operator $R_Q : L_p(Q) \to L_p(Q)$ is given by the formula $R_Q = P_Q R I_Q$, where

$$Ru(x) = \sum_{h \in \mathcal{M}} a_h u(x + h),$$

$a_h \in \mathbb{R}$, $\mathcal{M} \subset \mathbb{Z}^n$ is finite set of vectors, $I_Q$ is the extension operator of functions from $L_p(Q)$ by zero in $\mathbb{R}^n \setminus Q$, and $P_Q$ is the restriction operator of functions from $L_p(\mathbb{R}^n)$ to $Q$.

It is well known that problem (1), (2) without perturbation (i.e., for $R \equiv \mathbb{I}$) has a unique solution. In a linear case (i.e., for $p = 2$), the existence and uniqueness of a generalized solution of strongly elliptic differential-difference equation (1) with boundary condition (2) was proved in [1]. We formulate sufficient existence conditions for a generalized solution of nonlinear problem (1), (2) for any $p \in (2, \infty)$ and $f_0 \in L_q(Q)$. We also construct examples, which illustrate the distinction between linear and nonlinear cases.

The research was supported by RFBR grant 09-01-00586.

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# The Existence and $C^1$-Smoothness of Local Center-Unstable Manifolds for Differential Equations with State-Dependent Delay

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We consider a functional-differential equation

$$\dot{x}(t) = f(x_t)$$

with a function $f$ defined on an open subset of $C^1([-h, 0], \mathbb{R}^n)$, $h > 0$, and taking values in $\mathbb{R}^n$. Under mild smoothness conditions on $f$, the functional-differential equation above induces a semiflow on a submanifold of $C^1([-h, 0], \mathbb{R}^n)$. In particular, the imposed smoothness assumptions are often fulfilled if $f$ represents the right-hand part of a differential equation with state-dependent delay. We discuss a result proving that the semiflow of the functional-differential equation has continuously differentiable local center-unstable manifolds at a stationary point. Additionally, we give an example of a differential equation with a state-dependent delay, where the existence of such local center-unstable manifolds may be used to prove the existence of a periodic solution.

# Isoperimetric Problems in Three-Dimensional Homogeneous Spaces and Integrable Systems

I. A. Taimanov
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We discuss applications of integrable systems theory to the constructing of constant mean-curvature surfaces in three-dimensional homogeneous manifolds different from space forms.

# The Gårding-Type Inequality for Some Class of Functional-Differential Equations

A. L. Tasevich
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We consider a functional-differential equation with the Dirichlet conditions and with contractions and expansions of arguments:

$$A_R u(x) = \sum_{|\alpha|,|\beta| \leqslant m} D^\alpha R_{\alpha\beta} D^\beta u(x) \quad (x \in \Omega), \tag{1}$$

$$R_{\alpha\beta}v(x) = \sum_{j=-k}^{k} a_{\alpha\beta j}v(q^{-j}x),$$

where $q > 1$ is an arbitrary fixed real number, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary and such that $\overline{\Omega} \subset q\Omega$, and the coefficients $a_{\alpha\beta j}(x) \in C^\infty(\overline{\Omega})$ are given complex-valued functions. The functions are supposed to be extended by zero outside $\Omega$ before applying $R_{\alpha\beta}$ to them. We obtain a sufficient condition, under which the Gårding-type inequality

$$\operatorname{Re}(A_R u, u)_{L_2(\Omega)} \geqslant c_1\|u\|^2_{H^m(\Omega)} - c_2\|u\|^2_{L_2(\Omega)} \quad (u \in C_0^\infty(\Omega))$$

holds. The significance of this inequality lies in the fact that together with the compact embedding $\overset{\circ}{H}{}^m(\Omega) \subset L_2(\Omega)$ it guarantees the Fredholm solvability and discreteness and semiboundedness of the spectrum of Eq. (1) with the Dirichlet boundary condition.

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# Periodic Solutions of Parabolic Problems with Discontinuous Hysteresis


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As a prototype model, we consider the heat equation with hysteresis feedback control on the boundary

$$u_t(x,t) = \Delta u(x,t), \qquad x \in Q, \ t > 0,$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial Q} = H(u)(x,t), \qquad x \in \partial Q, \ t > 0,$$

where $Q \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial Q$, $\nu$ is the outward normal to the boundary, and $H$ is a discontinuous (nonlinear) hysteresis operator of relay type. The model describes various processes of automatic thermal control. It was originally proposed in [1]. Since then, it was treated by many mathematicians, but the complete description of its long-time behavior remained an open question.

By reducing the problem to a discontinuous infinite-dimensional dynamical system, we suggest an algorithm which allows one to construct *all* periodic solutions with exactly two switchings on the period and study their stability.

We show that the following situations are possible depending on the characteristics of hysteresis $H$:

(1) There is a unique periodic solution. It is stable, and it is a global attractor.

(2) Several periodic solutions with different geometrical structure and different stability properties coexist.

Finally, we discuss mechanisms of appearance and disappearance of periodic solutions based on the discontinuous dynamics of hysteresis.

The talk is based on papers [2, 3].

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# Integrodifferential Equations in a Hilbert Space and Their Applications

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We study integrodifferential equations with unbounded operator coefficients in a Hilbert space

$$\frac{du(t)}{dt} + \int_0^t K(t-s)A^2u(s)ds = f(t), \quad t \in \mathbb{R}_+, \tag{1}$$

$$u(+0) = \varphi_0, \tag{2}$$

where $A$ is a self-adjoint positive operator with compact inverse acting on a Hilbert space $H$ and the kernel $K(t)$ is a scalar convex downwards decreasing function. Moreover, $K(t)$ belongs to the space $W_1^1(\mathbb{R}_+)$.

The equation (1) is an abstract form of the Gurtin–Pipkin integrodifferential equation, which describes the heat propagation in media with memory and sound propagation in viscoelastic media; it also arises in homogenization problems in porous media (Darcy law).

We obtain the results on correct solvability of problem (1), (2) in weighted Sobolev spaces on a positive semiaxis $\mathbb{R}_+$. Additionally, assuming that the kernel $K(t)$ has the form

$$K(t) = \sum_{j=1}^{\infty} \frac{c_j}{\gamma_j} e^{-\gamma_j t}, \tag{3}$$

where $c_j > 0$, $\gamma_{j+1} > \gamma_j > 0$, $j \in \mathbb{N}$, and $\gamma_j \to +\infty$ $(j \to +\infty)$, we provide the spectral analysis of the operator-function $L(\lambda)$, which is the symbol of the equation (1). Let $e_j$ denote the orthonormal basis composed of the eigenvectors of the operator $A$ corresponding to eigenvalues $e_j$, i.e., such that $Ae_j = a_j e_j$ for $j \in \mathbb{N}$. The eigenvalues $a_j$ are numbered in increasing order: $0 < a_1 < a_2 < \ldots$; an $a_n \to +\infty$ as $n \to +\infty$. The spectrum of the operator-function $L(\lambda)$ can be represented in the form

$$\sigma(L) := \overline{(\cup_{n=1}^{\infty} \cup_{k=1}^{\infty} \lambda_{kn}) \cup \left(\cup_{n=1}^{\infty} \lambda_n^{\pm}\right)}, \tag{4}$$

where $\{\lambda_{k,n}|k \in \mathbb{N}\}$ is a countable series of real zeros lying on the negative semiaxis and $\lambda_n^{\pm}$ is a pair of complex conjugate zeros lying in the left half-plane such that $\lambda_n^+ = \overline{\lambda_n^-}$ of the meromorphic function $l_n(\lambda) := (L(\lambda)e_n, e_n)$. On the base of spectral analysis, we obtain the representation of the solution of problem (1), (2) as a series of exponents corresponding to the eigenvalues of the operator-function $L(\lambda)$.

The detailed statements of the problems, formulations, and proofs of the results can be found in [1–3].

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# The Sunflower Equation


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We consider the delay equation:

$$\varepsilon x''(t) + \alpha x'(t) + \beta \sin(x(t-\varepsilon)) = 0$$

with $\alpha, \beta, \varepsilon > 0$ on the space $C := \{f : [-\varepsilon, 0] \to \mathbb{R}^2, \text{ continous}\}$.

This delay equation describes the motion of the sunflower, where $x(t)$ is the angle of the plant with the vertical, $\varepsilon$ is the time lag depending on the geotropic reaction of the sunflower, which is caused by the auxin concentration in the plant, and $\alpha$ and $\beta$ are fixed real constants.

We want to study the global dynamical properties of this delay equation. In particular, we are interested to find a global attractor.

For technical details, we refer the audience to the work of Marcos Lizana.

# A Simple Market Model


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We consider a caricature of a market, where at time $t$ there are $b_i$ buyers prepared to buy a unit of a commodity at a price $c_i$ and $s_i$ sellers prepared to sell at this price, $c_{i-1} < c_i$, $1 < i \leqslant N$. A seller who did not get the trade at the price $c_i$ with some probability moves to the price level $c_{i-1}$, while a buyer who did not get the trade moves the price level $c_{i+1}$. Presuming that the number of traders is $\gg 1$, we come to the following system of ODE:

$$\begin{array}{l} \dot{b}_1 = \lambda_b - \alpha_b b_1 - \gamma \min\, \big[b_1, s_1\big], \\ \dot{b}_i = \alpha_b b_{i-1} - \alpha_b b_i - \gamma \min\, \big[b_i, s_i\big],\ 1 < i \leqslant N, \\ \dot{s}_i = \alpha_s s_{i+1} - \alpha_s s_i - \gamma \min\, \big[b_i, s_i\big],\ 1 \leqslant i < N, \\ \dot{s}_N = \lambda_s - \alpha_s s_N - \gamma \min\, \big[b_N, s_N\big], \end{array} \tag{1}$$

$$b_i(0) = \tilde{b}_i \geqslant 0, \quad s_i(0) = \tilde{s}_i \geqslant 0. \tag{2}$$

Here $\lambda_{b/s} > 0$, $\alpha_{b/s}, \gamma$, and $0 < \alpha_{b/s}, \gamma < 1$ are parameters.

**Statement 1.** *A solution of* (1), (2) *exists for all* $t > 0$ *and it tends to the single stationary solution of* (1) *as* $t \to \infty$.

A solution of (1), (2) obeys a kind of *min/max principle*. For example, if $\{b'_i\}, \{s'_i\}$ and $\{b''_i\}, \{s''_i\}$ are solutions of (1), (2), where

$$b'_i(0) < b''_i(0), \quad s'_i(0) > s''_i(0),$$

then for $t > 0$,

$$b'_i(t) < b''_i(t), \quad s'_i(t) > s''_i(t).$$

If $N$ is large, $N \to \infty$, then we consider a system

$$\begin{array}{l} \dot{b}(0,t) = \lambda_b - \alpha'_b \dfrac{db(0,t)}{dx} - \gamma \min\, \big[b(0,t), s(0,t)\big], \\ \dot{x}(x,t) = -\alpha_b \dfrac{d\ b(x,t)}{dt} - \gamma \min\, \big[b(x,t), s(x,t)\big], \\ \dot{s}(x,t) = \alpha_s \dfrac{d\ s(s,t)}{dt} - \gamma \min\, \big[b(x,t), s(x,t)\big], \\ \dot{s}(1,t) = \lambda_s + \alpha'_s \dfrac{ds(1,t)}{dx} - \gamma \min\, \big[b(1,t), s(1,t)\big], \\ 0 < x \leqslant 1, \end{array} \tag{3}$$

$$b(x,0) = \tilde{b}(x), \quad s(x,0) = \tilde{s}(x). \tag{4}$$

For solution of (3), (4), a statement similar to Statement 1 is valid.

# On the Generalized Riemann–Hilbert Problem for Monodromy Data of a Scalar Equation

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The classical Riemann–Hilbert problem, i. e., the question on existence of a *Fuchsian* system

$$\frac{dy}{dz} = \sum_{i=1}^{n} \frac{B_i}{z - a_i}\, y, \qquad y(z) \in \mathbb{C}^p, \qquad B_i \in \mathrm{Mat}(p, \mathbb{C}),$$

of $p$ linear differential equations with prescribed singularities $a_1, \ldots, a_n \in \mathbb{C}$ and monodromy

$$\chi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_n\}, z_0) \longrightarrow \mathrm{GL}(p, \mathbb{C}), \tag{1}$$

has a negative answer in the general case, as was shown by A. A. Bolibrukh. He also obtained various sufficient conditions for the positive solution of the problem, one of which is the following (see [1]).

**Condition 1.** *If the representation* (1) *is the monodromy of a linear differential equation*

$$\frac{d^p u}{dz^p} + b_1(z)\frac{d^{p-1}u}{dz^{p-1}} + \ldots + b_p(z)u = 0 \tag{2}$$

*of order p such that all its singularities $a_1, \ldots, a_n$ are* Fuchsian*, then the Riemann–Hilbert problem has a positive solution.*

Here we consider the *generalized* Riemann–Hilbert problem for linear systems with *irregular* (non-Fuchsian) singular points, i. e., the question on existence of a system of $p$ linear differential equations with prescribed singularities $a_1, \ldots, a_n \in \mathbb{C}$ and prescribed generalized monodromy data. By the latter, one means the monodromy representation (1) and local meromorphic invariants at each singular point (see [2]). Furthermore, one requires the coefficient matrix of a system to have a pole of minimal order at each singular point, i. e., the pole order cannot be reduced by a local meromorphic transformation $\tilde{y} = \Gamma(z)\, y$ of the dependant variable. The main result is the following analogue of the sufficient Condition 1 for the problem under consideration.

**Theorem.** *The generalized Riemann–Hilbert problem for the generalized monodromy data of a scalar equation* (2) *such that all ita singularities are formally unramified, has a positive solution.*

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# On the Linearization Problem for Neutral Equations with State-Dependent Delays


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Neutral functional-differential equations of the form

$$x'(t) = g(\partial\, x_t, x_t)$$

define continuous semiflows $G$ on closed subsets in manifolds of $C^2$-functions under hypotheses designed for the application to equations with state-dependent delay. The differentiability of the solution operators $G(t, \cdot)$ in the usual sense is not available, but for a certain variational equation along flowlines, the initial value is well-posed. Using this variational equation, we prove a principle of linearized stability, which covers the prototype

$$x'(t) = A(x'(t + d(x(t)))) + f(x(t + r(x(t))))$$

with nonlinear real functions $A$, $d < 0$, $f$, and $r \leqslant 0$. Special cases of the latter describe the interaction of two kinds of behaviour, namely, following a trend versus negative feedback with respect to a stationary state.

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# Stability Properties of Equilibria and Periodic Solutions in Systems with Large Delay

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Systems of delay-differential equations (DDEs) with large delay represent a special class of singularly perturbed problems and can exhibit dynamics on different time scales. Linearizing a DDE with large delay at an equilibrium point, we obtain a spectrum that splits into two different parts. One part called the strong spectrum converges to isolated points as the delay parameter tends to infinity. The other part called the pseudo-continuous spectrum accumulates near the criticality and converges after rescaling to a set of spectral curves called the asymptotic continuous spectrum. In both cases, the limiting spectra can be approximated by the spectrum of a suitable scale free problem. In this talk, we show how these results can be generalized to the Floquet spectrum of certain periodic solutions of DDEs with large delay.

# State-Dependent Delay Differential Equations for Subspace Clustering

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We developed a neural network architecture — projective adaptive resonance theory (PART) — to detect low-dimensional patterns in a high-dimensional data set. This theory has been applied for gene filtering and cancer diagnosis, neural spiking trains clustering, ontology construction, and stock associations detection. The key feature of a PART network is a hidden layer which incorporates a selective output signal mechanism (SOS) that calculates the similarity between the output of a given input neuron with the corresponding component of the template of a candidate cluster neuron and allows the signal to be transmitted to the cluster neuron only when the similarity measure is sufficiently large. This clustering architecture was recently refined to incorporate adaptive transmission delays and signal transmission

information loss (PART-D). The resultant selective SOS is based on the assumption that the signal transmission velocity between input processing neurons and clustering neurons is proportional to the similarity between the input pattern and the feature vector of the clustering neuron. The mathematical model governing the evolution of the signal transmission delay (the short-term memory traces and the long-term memory traces) represents a new class of delay differential equations, where the evolution of the delay is described by a nonlinear differential equation involving the aforementioned similarity measure. This talk will describe the PART-D architecture and the associated delay differential systems, and discuss future directions how state-dependent delay differential equations can be used to design algorithms for clustering in skewed subspaces or sub-manifolds of the data space.

# Periodic and Relative Periodic Solutions in Systems with Time-Delay

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In my talk, I discuss the existence and stability properties of periodic solutions in delay differential equations with fixed delay. If delay is considered as a parameter, then the number of coexisting periodic solutions grows linearly with the increasing of the delay. Similar phenomenon takes place also in delay differential equations equivariant with respect to the $S^1$ symmetry. In this case, instead of periodic solutions, the number of quasiperiodic (relative periodic) solutions increases with the delay. I will make some estimations on the number of such solutions and show an example of a laser model.

# Homogenization of Navier–Stokes Systems for Electro-Rheological Fluid

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Some fluids sharply change their rheological properties in the presence of an electromagnetic field. The viscous stress tensor not merely becomes a nonlinear function of the deformation velocity tensor $D$; it begins to essentially depend on the spatial argument $x$. An example is the tensor $|D|^{p(x)-2}D$, where the variable exponent is determined by the applied electromagnetic field; usually, the viscous stress tensor has a more complicated anisotropic and non-variational structure. The mathematical theory of electro-rheological fluids is exposed in the book [1].

The electromagnetic field can be assumed to be periodic in $x$ with small period. In this case, the homogenization problem arises; our main interest is to find a homogenized (effective) viscous stress tensor, which must be independent of the spatial variable due to the homogenization.

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# Monodromy Operator Approximation for Periodic Solutions of Differential-Difference Equations

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The following nonlinear equation with delay is considered:

$$x'(t) = f(x(t), x(t-1)). \tag{1}$$

It is supposed that a periodic solution $\tilde{x}$ of this equation with period $T$ is known. The Floquet theory is known to be valid for Eq. [1], which permits to describe the behavior of the solutions in the field the of periodical solution $\tilde{x}$ in terms of monodromy operator eigenvalues associated with the $\tilde{x}$ solution.

Unlike the case of ordinary differential equations, yet there is no a universal efficient method to find Floquet multipliers for the casual Eq. (1) and solution $\tilde{x}$ (at least approximately). In [2–4], there are most sufficient results for such an approach. If the period is not commensurable with delay, then considerable difficulties arise. One of the ways to solve this problem is based on the approximation of an original problem by means of a sequence of auxiliary problems. The problem of building of auxiliary model problems has not been solved yet and still there is no evaluation method to calculate the error within such an approach to the finding of Floquet multipliers.

A new way of approximation of the original problem is introduced in this paper. A number of examples are provided.

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# Optimal Feedback Control in A Stationary Mathematical Model for the Motion of Polymers

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We will consider the weak formulation of the optimal feedback control problem for the initial-boundary problem

$$\sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \nu \Delta v - \varkappa Div \left( v_k \frac{\partial \mathcal{E}(v)}{\partial x_k} \right) + gradp = f \in \Psi(v), \quad x \in \Omega; \tag{1}$$

$$divv = 0, \quad x \in \Omega; \tag{2}$$

$$v|_{\partial\Omega} = 0, \tag{3}$$

where $v$ is the vector function of velocities at points of the domain $\Omega$ in the space $\mathbb{R}^n, n = 2, 3$, with the boundary $\partial\Omega$, $p$ is the pressure function, $\mathcal{E}(v) = (\mathcal{E}_{ij}) = (\frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}))$ is the strain rate tensor, $f$ is the density of external forces, $\nu$ is the kinematic coefficient of viscosity, and $\varkappa$ is the delay time. The coefficient $\varkappa$ is also called the relaxation time of deformation.

Let $\Psi : V \multimap V^*$ satisfy the following conditions:

($i_1$) its values are nonempty, compact, and convex;

($i_2$) it is semicontinuous and compact;

($i_3$) it is globally bounded;

($i_4$) it is weakly closed.

Let $\mathcal{V} = \{v \in C^\infty(\Omega)^n,\ divv = 0\}$, $V$ be the closure of $\mathcal{V}$ in the norm of the space $W_2^1(\Omega)^n$, and $X$ be the closure of $\mathcal{V}$ in the norm of the space $W_2^3(\Omega)^n$.

**Theorem 1.** *Let the mapping* $\Psi$ *satisfy* ($i_1$)–($i_4$). *Then the boundary-value problem* (1)–(3) *has at least one weak solution.*

Let $\Sigma \subset V \times V^*$ be the set of weak solutions of problem (1)–(3). We will consider an arbitrary cost functional $\Phi : \Sigma \to \mathbb{R}$ satisfying the following conditions:

($j_1$) there exists a number $\gamma$ that $\Phi(v, f) \geqslant \gamma$ for all $(v, f) \in \Sigma$;

($j_2$) if $v_m \rightharpoonup v_*$ in $V$ and $f_m \to f_*$ in $V^*$, then $\Phi(v_*, f_*) \leqslant \varliminf_{m\to\infty} \Phi(v_m, f_m)$.

**Theorem 2.** *Let the mapping* $\Psi$ *satisfies* ($i_1$)–($i_4$) *and the functional* $\Phi$ *satisfies* ($j_1$) *and* ($j_2$). *Then the boundary-value problem* (1)–(3) *has at least one weak solution* $(v_*, f_*)$ *such that* $\Phi(v_*, f_*) = \inf_{(v,f)\in\Sigma} \Phi(v, f)$.

# Trajectory and Global Attractors for Equations of Non-Newtonian Hydrodynamics

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It is known that the theory of attractors describes behavior of systems over time. This behavior (so-called "limit regimes") frequently characterizes the process as a whole. The theory of attractors of dynamical systems has been thoroughly developed and has numerous applications in the study of ordinary differential equations. This theory was generalized by O. A. Ladyzhenskaya for the case of the two-dimensional Navier–Stokes system, which describes flat flows. Yet the three-dimensional Navier–Stokes system demanded a new approach owing to the absence of results concerning either the uniqueness of weak solutions or the existence of global strong ones. Such an approach based on new concepts of trajectory attractors was suggested in [1, 2]. This theory was applied by V. V. Chepyzhov and M. I. Vishik in order to prove the existence of a trajectory and global attractors for the three-dimensional Navier–Stokes system.

However, the results of existence of trajectory attractors rests solidly on the translation invariance of the trajectory space. Yet in the case of more complex equations of non-Newton hydrodynamics translation invariant trajectory spaces were not found; for this reason, a theory of trajectory attractors that does not need the trajectory space to be translation invariant was developed in [3]. In [4], it was proved that this theory is useful in order to establish the existence of a trajectory and global attractors for the Jeffreys model of viscoelastic medium.

The report proposes to present:

(1) a general scheme to prove the existence of a trajectory and global attractors for the spaces non-invariant under shifts of the trajectory;

(2) a proof of the existence of the trajectory and global attractors for the model with the objective derivative;

(3) a visualization of the attractor of the perturbed Poiseuille flow (a visualization of the attractor means a way to represent them graphically).

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# Роль центральных элементов в построении уравнений типа Книжника—Замолодчикова

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Доклад будет посвящен многомерному обобщению классической проблемы Римана—Гильберта построения системы Пфаффа типа Фукса с сингулярным дивизором, определяемым набором гиперплоскостей отражения одной из классических систем корней. При построении уравнений Книжника—Замолодчикова, ассоциированных с системой корней $A_n$, ранее использовался элемент Казимира второго порядка универсальной обертывающей алгебры соответствующей алгебры Ли. Для других систем корней такие построения практически отсутствуют. В случае системы корней типа $B_n$ аналогичное построение было получено А. Лейбманом в однопараметрическом случае, в то время как уравнение должно зависеть от двух параметров — по числу орбит соответствующей системы корней. В докладе будет дана конструкция систем Пфаффа типа Фукса (аналогов уравнений Книжника—Замолодчикова) для различных систем корней, основанная на центральных элементах соответствующей обертывающей алгебры Ли порядка большего двух.

# Нелинейные дискретные уравнения типа свертки с ядрами специального вида

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В монографии [1] рассмотрены основные классы нелинейных дискретных уравнений типа свертки

$$F(n, u_n) + \sum_{k=-\infty}^{\infty} h_{n-k} u_k = f_n,$$

$$u_n + \sum_{k=-\infty}^{\infty} h_{n-k} F(k, u_k) = f_n,$$

$$u_n + F\left(n, \sum_{k=-\infty}^{\infty} h_{n-k} u_k\right) = f_n$$

в вещественных и комплексных пространствах $\ell_p$, $1 < p < \infty$. Получены теоремы о существовании, единственности и способах нахождения решений указанных уравнений. В настоящей работе, в случае ядер специального вида, дано дополнение и усиление некоторых результатов, касающихся, в частности, приближенного решения соответствующих нелинейных дискретных уравнений.

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# Исследование линейных дифференциальных уравнений и операторов с неограниченными операторными коэффициентами методами спектральной теории разностных операторов и линейных отношений

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Многие свойства решений (ограниченность, почти периодичность, устойчивость) линейных дифференциальных уравнений с неограниченными операторными коэффициентами тесно связаны с соответствующими свойствами дифференциального оператора, определяющего рассматриваемое уравнение и действующего в подходящем функциональном пространстве. Его свойства обратимости, корректности, фредгольмовости, структура спектра зависят от размерности ядра, коразмерности образа, их дополняемости. Вводится понятие состояния линейного отношения (многозначного линейного оператора), с которым ассоциируется совокупность свойств его ядра и образа. Изучаемому дифференциальному оператору (соответствующему уравнению) сопоставляется линейный разностный оператор (разностное отношение) и доказываются утверждения о совпадении множества их состояний, необходимые и достаточные условия их фредгольмовости. При доказательстве основных результатов существенно используется свойство экспоненциальной дихотомии семейства эволюционных операторов и спектральная теория линейных отношений. Делается обзор результатов об оценке норм обратных операторов, о применении метода подобных операторов к исследованию дифференциальных операторов, спектральной теории дифференциальных операторов в весовых пространствах функций.

# Об одной эллиптической краевой задаче

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В плоской односвязной области $G$ с кусочно $C^{3,\alpha}$-гладкой границей $\Gamma$, $\alpha \in (0,1)$, рассматривается однородная задача Дирихле ($u(x) = 0$, $x \in \Gamma$) для уравнения Гельмгольца $\Delta u(x) - au(x) = b$, где $-a$ не совпадает с каким-либо собственным числом $\lambda_k$ оператора Лапласа, дополненная нелокальным условием $\int_\Gamma \partial_\nu u ds = 1$, где $ds$ — элемент длины дуги $\Gamma$, а $\partial_\nu$ — производная по внешней нормали к $\Gamma$. Последнее условие связывает параметры $a$ и $b$ зависимостью $b(a) = \left[|G| - a \int_G \int_G \mathcal{G}_a(x, y) dx dy\right]^{-1}$, где $\mathcal{G}_a(x, y)$ — функция Грина, и, тем самым, делает рассматриваемую задачу зависящей лишь от параметра $a$. Такая задача изучалась, в частности, в [1] в связи с физическими приложениями. С использованием результатов из [1] установлены асимптотики

$$b(a) = |\Gamma|^{-1}\sqrt{a} + \pi|\Gamma|^{-2} + O\left(a^{-1/2}\right), \quad \frac{d}{da}b(a) = \left(2|\Gamma|\sqrt{a}\right)^{-1} + O\left(a^{-3/2}\right), \quad a \to \infty,$$

а также асимптотика для решения $u(x) = u(x,a)$ внутри $G$:

$$au(x,a) + b(a) = \frac{\sqrt{a}}{|\Gamma|}e^{-r(x)\sqrt{a}}\left[A(x) + O\left(a^{-1/2}\right)\right], \qquad a \to \infty,$$

где $r(x)$ — расстояние от $x \in G$ до границы $\Gamma$, а гладкая функция $A(x)$ обращается в единицу при $x \in \Gamma$; во всех точках $x$ гладкости $\Gamma$ с кривизной $k(x)$ получены следующие асимптотики:

$$\partial_\nu u(x,\, a) = |\Gamma|^{-1} + \left[\pi|\Gamma|^{-2} - k(x)\left(2|\Gamma|\right)^{-1}\right]a^{-1/2} + O\left(a^{-1}\right), \qquad a \to \infty;$$

$$\frac{d}{da}\partial_\nu u(x,\, a) = \left[k(x)\left(4|\Gamma|\right)^{-1} - \pi 2^{-1}|\Gamma|^{-2}\right]a^{-3/2} + O\left(a^{-2}\right), \qquad a \to \infty.$$

Для указанной задачи с исключенным нелокальным условием (в этом случае краевая задача зависит от двух параметров, $a$ и $b$) установлено, что для положительности $au(x) + b$ необходимо и достаточно, чтобы $b > 0$ и $a > -\lambda_1$. Отметим, что необходимость этих условий была известна.

Работа выполнена при финансовой поддержке РФФИ (проект №10-01-00837), Программы ОМН РАН «Современные проблемы теоретической математики», проект «Оптимальные алгоритмы решения задач математической физики» и Программы №3 фундаментальных исследований ОМН РАН.

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# Нелокальные интегралы и квантование $N$ частиц

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Принцип дополнительности Н. Бора привел к квантованию физических величин в квантовой механике, к вторичному квантованию и к развитию теории поля, в частности, в работах Н.Н. Боголюбова и его школы. Речь идет о классических гамильтоновых системах в окрестности минимума гамильтониана. Традиционно минимум отвечает стационарной точке в фазовом пространстве. Мы объясняем, каким образом точку можно заменить на нетривиальную периодическую орбиту.

Рассмотрим гамильтоново полиномиальное степени $n$ векторное поле на плоскости. Зафиксируем выпуклый (для простоты) овал гамильтониана. Во внешности этого овала выберем $k$ различных овалов, гомологичных исходному и, соответственно, $k$ точек — по одной на каждой кривой. Тогда найдутся $N$ функций $f_i(x)$, $N$ однопараметрических подгрупп диффеоморфизмов на $N$ различных фазовых кривых, так что

$$\sum_{i=1}^{N} f_i\left(x|_{g_i(t)}\right) \equiv Const, \tag{1}$$

т. е. является первым интегралом движения, причем $N \geqslant k \cdot n$ и носитель интеграла (1) может возвращаться в исходное положение.

**Теорема.** *В случае общего положения время возвращения оценивается величиной $Const^N$ и требует квантования указанных $N$ орбит.*

**Замечание.** Возможно, этот факт служит причиной отсутствия указаний в литературе на существование нелокальных интегралов и их значение для задач математической физики, в частности, квантовой механики.

# Исследование слабого решения задачи о возбуждении электромагнитных колебаний в области с киральным заполнением

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Настоящая работа посвящена исследованию задачи о возбуждении электромагнитных колебаний заданным локальным распределением зарядов и токов в области с неоднородным киральным заполнением, характеризующимся материальными уравнениями $\mathbf{D} = \varepsilon\mathbf{E} + i\xi\mathbf{B}$, $\mathbf{H} = i\xi\mathbf{E} + \mu^{-1}\mathbf{B}$.

Область $\Omega$, в которой рассматривается задача, может быть либо конечной с регулярной идеально проводящей границей $\Gamma$, либо представлять собой дополнение к ограниченной области $\tilde{\Omega}$ с регулярной идеально проводящей границей $\Gamma$. Пусть область $\Omega$ состоит из конечного числа подобластей: $\Omega = \cup_{k=0}^{q}\Omega_k$, причем все из них, кроме быть может, подобласти $\Omega_0$, конечны, и общая для подобластей $\Omega_k$ и $\Omega_m$ граница $\Gamma_{km}$ регулярна и ограничена. Подобласть $\Omega_0$ неограничена в случае бесконечной области $\Omega$. Пусть подобласти $\Omega_k$ имеют однородные киральные заполнения с параметрами: $\varepsilon = \varepsilon_k > 0$, $\mu = \mu_k > 0$, $\xi = \xi_k \geqslant 0$, $\sigma = \sigma_k \geqslant 0$, где $k = \overline{0, q}$, причем $\xi_0 = 0$ и $\sigma_0 = 0$, если подобласть $\Omega_0$ неограничена.

Если в области $\Omega$ имеются сторонние токи плотности $\mathbf{j} \in L_2(0, T; \Omega)$, то начально-краевая задача для векторов $\{\mathbf{E}, \mathbf{H}\} = \left\{\mathbf{E}^k, \mathbf{H}^k\right\}$ в $\Omega_k$, $k = \overline{0, q}$, имеет вид

$$\begin{cases} (\varepsilon_k + \xi_k^2\mu_k)\frac{\partial \mathbf{E}^k}{\partial t} + i\xi_k\mu_k\frac{\partial \mathbf{H}^k}{\partial t} - rot\mathbf{H}^k + \sigma_k\mathbf{E}^k = -\mathbf{j}, \quad \Omega_k \times (0, T], \\ -i\xi_k\mu_k\frac{\partial \mathbf{E}^k}{\partial t} + \mu_k\frac{\partial \mathbf{H}^k}{\partial t} + rot\mathbf{E}^k = 0, \quad \Omega_k \times (0, T], \\ [\mathbf{E}^k, \mathbf{n}]\big|_{\Sigma_{km}} = [\mathbf{E}^m, \mathbf{n}]|_{\Sigma_{km}}, \quad [\mathbf{E}^k, \mathbf{n}]\big|_{\Sigma_{km}} = [\mathbf{E}^m, \mathbf{n}]|_{\Sigma_{km}}, \quad [\mathbf{E}^0, \mathbf{n}]\big|_{\Gamma} = 0, \\ \mathbf{E}|_{t=0} = \mathbf{E}_0, \quad \mathbf{H}|_{t=0} = \mathbf{H}_0. \end{cases} \tag{1}$$

На основании метода Галеркина доказано существование единственного обобщенного решения задачи (1) при условии, что компоненты векторов $\mathbf{E}_0$ и $\mathbf{H}_0$ и их роторов принадлежат пространству $L_2(0,T;\Omega)$.


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## $H$-теорема для дискретных квантовых кинетических уравнений и их обобщений


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Доказывается $H$-теорема для обобщений уравнений химической кинетики. Важными физическими примерами такого обобщения являются дискретные модели квантовых кинетических уравнений (уравнений Улинга—Уленбека) и квантовый марковский процесс (квантовое случайное блуждание). Доказывается совпадение временных средних с экстремалями по Больцману для уравнений Лиувилля и их обобщений.

Рассмотрим систему уравнений:

$$\frac{df_i}{dt} = \sum_{(\alpha,\beta)\in\Im} (\beta_i - \alpha_i)\,\sigma_\beta^\alpha(\mathbf{f})\,\tilde{K}_\beta^\alpha e^{(\alpha,\nabla G(\mathbf{f}))}, \quad i = 1,2,\ldots,n, \tag{1}$$

где $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ и $\beta = (\beta_1,\beta_2,\ldots,\beta_n)$ — векторы с целочисленными неотрицательными компонентами, а суммирование ведется по конечному множеству $\Im$ мультииндексов $(\alpha,\beta)$, симметричному относительно перестановок $\alpha$ и $\beta$; $\sigma_\beta^\alpha(\mathbf{f})$, $G(\mathbf{f})$ — заданные функции от $\mathbf{f} = (f_1,f_2,\ldots,f_n)$, $\sigma_\beta^\alpha(\mathbf{f}) = \sigma_\alpha^\beta(\mathbf{f}) > 0$; $\tilde{K}_\beta^\alpha \geqslant 0$.

Если $\sigma_\beta^\alpha(\mathbf{f})$ не зависит от $\mathbf{f}$, то мы имеем систему

$$\frac{df_i}{dt} = \sum_{(\alpha,\beta)\in\Im} (\beta_i - \alpha_i)\,K_\beta^\alpha e^{(\alpha,\nabla G(\mathbf{f}))}, \quad i = 1,2,\ldots,n, \quad K_\beta^\alpha = \sigma_\beta^\alpha \tilde{K}_\beta^\alpha. \tag{2}$$

Если в (2) $\frac{\partial G(\mathbf{f})}{\partial f_i} = \ln f_i$, то мы получаем систему уравнений химической кинетики. В химической кинетике рассматриваются условие детального равновесия и условие динамического равновесия (условие Штюккельберга—Батищевой—Пирогова). Мы формулируем обобщение условия детального равновесия для систем (1) и обобщение условия Штюккельберга—Батищевой—Пирогова для систем (2) и доказываем в этих случаях $H$-теорему, причем получается, что $H(\mathbf{f}) = G(\mathbf{f}) - (\nabla G(\xi),\mathbf{f})$.

Работа выполнена при финансовой поддержке РФФИ (код проекта 11-01-00012) и программы Отделения математических наук РАН 3.5.

## Составление модели движения планетохода с циклическим контактом с поверхностью

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Представляются характеристика и рекомендации для развития методов передвижения планетохода при пониженной гравитации небесного тела относительно Земной, а также вариант исполнения такого типа аппаратов.

Управление движением планетохода с циклическим контактом с поверхностью выполняют за счет вращательного движения колесной пары и колебательного движения каретки с пассажирским (грузовым) модулем путем создания дополнительной прижимной силы упругой колесной пары к поверхности за счет вертикальной составляющей центробежной силы движения каретки и совмещения его с упругими колебаниями колесной пары.

Разработана математическая модель передвижения аппарата как упругого колеса с маятниковым движением массы с осевой подвеской. Движение аппарата рассматривается в плоскости. Планетоход представляется как механическая система с тремя степенями свободы, имеющая колесную пару радиусом $l_1$, движущуюся по поверхности и маятник, подвешенный в оси колесной пары на расстоянии $l_2$. Колесная пара имеет массу $m_1$ и момент инерции $I_1$; масса маятника $m_2$, момент инерции $I_2$. Между колесной парой и маятником действует возмущающий момент $M$.

Разработана математическая модель взаимодействия упругой колесной пары с поверхностью в зависимости от глубины проседания. Поверхность абсолютно гладкая.

Производится выбор проектных параметров аппарата и циклограммы подачи управляющего момента $M$ для выявления наиболее энергетически выгодных режимов движения.

Показаны полученные траектории прыжкового движения и их энергетические преимущества для определенного диапазона скоростей движения.

## Параметрические представления псевдодифференциальных операторов и краевые задачи для дифференциальных уравнений электродинамики

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Речь идет о сведении 2D и 3D краевых задач для стационарных уравнений Максвелла на плоскопараллельных структурах к граничным гиперсингулярным интегральным уравнениям и их последующем приближенном решении. Центральный момент при выводе граничных интегральных уравнений — использование аналитического аппарата параметрических представлений псевдодифференциальных и интегральных операторов.

При приближенном решении полученных уравнений используются модификации численных методов дискретных особенностей. Найдены оценки скорости сходимости приближённых решений к точным. Построенные математические модели представляют практический интерес. В частности, с их помощью изучены задачи дифракции электромагнитных волн на предканторовых решетках и предфрактальных коврах Серпинского, которые используются в современной антенной технике.

# О разрешимости абстрактного дифференциального уравнения дробного порядка с переменным оператором

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В банаховом пространстве $E$ рассмотрим задачу типа Коши

$$D^{\alpha}_{\tau,t}u(t) = A(t)u(t), \quad t \in (\tau, b],\ \tau \geqslant 0, \tag{1}$$

$$\lim_{t\to\tau} D^{\alpha-1}_{\tau,t}u(t) = u_0, \tag{2}$$

где $D^{\alpha-1}_{\tau,t}u(t) = \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_{\tau}^{t}(t-s)^{-\alpha}u(s)\,ds$ — левосторонний дробный интеграл Римана—Лиувилля порядка $1-\alpha$, $0<\alpha<1$, $0\leqslant\tau\leqslant t$, $D^{\alpha}_{\tau,t}u(t) = \dfrac{d}{dt}I^{1-\alpha}_{\tau,t}u(t)$ — левосторонняя дробная производная Римана—Лиувилля порядка $\alpha$, $A(t)$ — линейный, замкнутый оператор.

**Условие 1.** *Линейный оператор $A(t)$ при каждом $t\in[0,b]$ имеет плотную в $E$ и не зависящую от $t$ область определения $D$; при любом $\lambda$ с $\operatorname{Re}\lambda\geqslant 0$ оператор $\lambda I - A(t)$ имеет ограниченный обратный, причем*

$$\left\|(\lambda I - A(t))^{-1}\right\| \leqslant \frac{M_1}{1+|\lambda|}, \quad M_1>0. \tag{3}$$

*Кроме того, для любых $t,\tau,s\in[0,b]$ справедливо неравенство*

$$\left\|(A(t)-A(\tau))A^{-1}(s)\right\| \leqslant M_2\ |t-\tau|^{\gamma}, \quad M_2>0,\ \gamma\in(0,1].$$

Из условия (3) вытекает, что оператор $A(t)$ при $\tau\geqslant 0$ является генератором сильно непрерывной полугруппы $e^{\tau A(t)}$ и

$$\left\|e^{\tau A(t)}\right\| \leqslant M_3\ e^{-\delta\tau}, \quad M_3>0,\ \delta>0.$$

**Теорема 1.** *Пусть выполнено условие* 1 *и* $u_0 \in D$. *Тогда задача* (1), (2) *имеет единственное решение.*

Доказательство разрешимости задачи (1), (2) проводится методом, развивающим метод Соболевского—Танабе [1].

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# Первые интегралы приводимых многомерных дифференциальных систем

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Рассматривается вполне разрешимая [1, с. 15–25] система в полных дифференциалах

$$dx = \sum_{j=1}^{m} A_j(t)\, x\, dt_j, \quad x \in \mathbb{R}^n, \quad t \in T \subset \mathbb{R}^m. \tag{1}$$

Одним из основных методов исследования линейных неавтономных систем является метод приведения их к линейным автономным системам с помощью той или иной группы преобразований [2,3]. Идея этого метода принадлежит А. М. Ляпунову [4], изучавшему обыкновенные неавтономные линейные системы, которые неособенными ограниченными линейными преобразованиями могут быть сведены к линейным системам с постоянными коэффициентами. Такие системы им были названы приводимыми. Существенное развитие теория обыкновенных приводимых систем получила в работе Н. П. Еругина [5]. В дальнейшем понятие приводимой обыкновенной дифференциальной системы было перенесено на многомерные дифференциальные системы (см. обзор литературы в [3]).

В данной работе по частным интегралам [1, с. 168–186] спектральным методом [6, 7] для системы (1), приводимой относительно некоторой группы нестационарных преобразований $G$ к автономной системе

$$dy = \sum_{j=1}^{m} B_j\, y\, dt_j, \quad y \in \mathbb{R}^n, \tag{2}$$

решена задача о построении интегрального базиса. Первые интегралы строятся в зависимости от кратности собственных чисел по общим собственным и присоединенным векторам матриц $C_j$, транспонированных к матрицам $B_j$, и по матрице преобразования $\mathrm{g} \in G$.

Так, например, в случае простых элементарных делителей имеет место

**Теорема.** *Пусть система* (1) *приводима к системе* (2) *с помощью матрицы преобразования* g, *а* $\nu$ — *общий собственный вектор матриц* $C_j$, *которому соответствуют собственные числа* $\lambda_j$, $j = \overline{1,m}$. *Тогда первыми интегралами системы* (1) *будут функции*

$$F_1\colon (t,x) \to \Big((\mathrm{Re}\,\nu\,\mathrm{g}(t)x)^2 + (\mathrm{Im}\,\nu\,\mathrm{g}(t)x)^2\Big)\exp\Big(-2\sum_{j=1}^{m}\mathrm{Re}\lambda_j\,t_j\Big)$$

$\forall (t,x) \in T \times \mathbb{R}^n$ *и*

$$F_2\colon (t,x) \to \mathrm{arctg}\,\frac{\mathrm{Im}\,\nu\,\mathrm{g}(t)x}{\mathrm{Re}\,\nu\,\mathrm{g}(t)x} - \sum_{j=1}^{m}\mathrm{Im}\,\lambda_j\,t_j$$

$\forall (t,x) \in \Lambda,\ \Lambda \subset \big\{(t,x)\colon t \in T,\ \mathrm{Re}\,\nu\,\mathrm{g}(t)x \neq 0\big\}$.

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# Неформальные решения ОДУ


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Рассмотрим обыкновенное дифференциальное уравнение

$$f(x,y,y',\dots,y^{(n)}) = 0, \tag{1}$$

где $f(x,y,y',\dots,y^{(n)})$ — это многочлен своих переменных. Пусть при $|x| \to 0$ ($\arg(x)$ ограничен с двух сторон) уравнение (1) имеет формальное решение

$$y = \sum c_s x^s,\ s \in \mathbf{K} \subset \mathbb{C}, \tag{2}$$

где $\mathbf{K} = \{s_0 + m_1 r_1 + m_2 r_2,\ m_1, m_2 \in \mathbb{Z},\ m_1 + m_2 \geqslant 0,\ m_1, m_2 \geqslant 0\}$, $s_0 \in \mathbb{C} \setminus \mathbb{Q}$, $r_1 = \langle R_1, (1, s_0)\rangle$, $r_2 = \langle R_2, (1, s_0)\rangle$, $R_1 = (\alpha_1, \beta_1)$, $R_2 = (\alpha_2, \beta_2)$, $R_1, R_2 \in \mathbb{Z}^2$,

$\mathrm{Re}\, s > \mathrm{Re}\, s_0$, показатели степени $s$ упорядочены по росту вещественных частей, $c_s$ – комплексные постоянные. Сделаем в уравнении (1) замену зависимой переменной

$$y = \sum_{s=s_0}^{s_m} c_s x^s + u, \tag{3}$$

где $m \in \mathbb{Z}_+$, $\mathrm{Re}(s_m - s_0) \geqslant n$, $s$ и $c_s$ — из формулы (2), после которой оно примет вид

$$f_0 \stackrel{\mathrm{def}}{=} \mathcal{L}(x)u + g(x, u, u', \ldots, u^{(n)}) = 0, \tag{4}$$

где линейный дифференциальный оператор $\mathcal{L}(x) = x^v \sum\limits_{l=1}^{n} a_l x^l \dfrac{d^l u}{dx^l}$, $\mathcal{L}(x) \not\equiv 0$, $v \in \mathbb{C}$, $a_l$ – комплексные постоянные; функция $g$ может содержать линейные по $u, u', \ldots, u^{(n)}$ члены вида $b_0 x^{v_1+m} \dfrac{d^m u}{dx^m}$ с $\mathrm{Re}\, v_1 > \mathrm{Re}\, v$, $v_1 \in \mathbb{C}$, $0 \leqslant m \leqslant n$, $b_0 = \mathrm{const} \in \mathbb{C}$, нелинейные по $u, u', \ldots, u^{(n)}$ члены и зависящие только от $x$ члены. Пусть в замене (3) $\mathrm{Re}\, s_m \geqslant \mathrm{Re}\, \lambda_i$, где $\lambda_i$, $i = 1, \ldots, n$, есть собственные значения оператора $\mathcal{L}(x)$. Тогда уравнение (4) имеет формальное решение

$$u = \sum_s c_s x^s, \tag{5}$$

где $\mathrm{Re}\, s \geqslant \mathrm{Re}\, s_{m+1} > n$, $\mathrm{Re}\, s_{m+1} > \mathrm{Re}\, \lambda_n$, $s \in \mathbf{K}$, $c_s$ — однозначно определенные комплексные постоянные.

**Теорема 1.** *Если в уравнении* (4), *которое получается из уравнения* (1) *после замены переменной* (3), *порядок старшей производной в* $\mathcal{L}(x)u$ *равен порядку старшей производной в сумме* $f_0$, *то ряд* (5) *сходится для достаточно малых* $|x|$.

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# О достаточных условиях стабилизации решения задачи Дирихле для параболического уравнения


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В цилиндре $D = Q \times (0, \infty)$, где $Q$ — область (возможно неограниченная) в $\mathbb{R}^N$, $N \geqslant 3$, рассмотрим задачу Дирихле

$$\begin{aligned} &Lu = \sum_{i,k=1}^{N} (a_{ik}(x,t) u_{x_k})_{x_i} - u_t = 0 \quad \text{в } D, \\ &u|_{t=0} = u_0(x), \quad x \in Q, \quad u|_S = 0 \end{aligned} \tag{1}$$

для равномерно параболического оператора $L$ с измеримыми ограниченными коэффициентами. Здесь $S = \partial Q \times (0, \infty)$ — боковая поверхность цилиндра $D$, $u_0(x)$ — ограниченная непрерывная в $Q$ функция, решение ограниченное и понимается в обобщенном смысле [1].

**Теорема 1.** *Если расходится интеграл*

$$\int^{\infty} \operatorname{cap}\left(\overline{B}_\tau \setminus Q\right) \cdot \tau^{1-N}\, d\tau = \infty, \tag{2}$$

*то решение задачи* (1) *имеет предел*

$$\lim_{t\to\infty} u(x,t) = 0 \tag{3}$$

*равномерно по $x$ на каждом компакте $K$ в $\mathbb{R}^N$.*

Здесь $\overline{B}_\tau = \{|x - x_0| \leqslant \tau\}$ — замкнутый шар с центром в произвольной точке $x_0$ радиуса $\tau$, $\operatorname{cap}(E)$ — винеровская емкость компакта $E \subset Q$.

Случай, когда коэффициенты в (1) не зависят от времени $t$, изучен в работе [2].

Имеет место следующее уточнение теоремы 1:

**Теорема 2.** *Если расходится интеграл* (2)*, то для решения задачи* (1) *справедлива оценка*

$$|u(x,t)| \leqslant C_1 \exp\left\{-C_2 \int_{a_0}^{\sqrt{t}} \operatorname{cap}\left(\overline{B}_\tau \setminus Q\right) \cdot \tau^{1-N}\, d\tau\right\},$$

*где $x$ — произвольная точка $Q$, $C_1 > 0$ — постоянная, зависящая от $N$ и постоянной эллиптичности $\lambda_1$, $C_2 > 0$ — постоянная, зависящая от $N$, $x$, $\lambda_1$.*

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# Три-ткани, определяемые системами ОДУ

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Рассматривается система обыкновенных дифференциальных уравнений

$$\frac{dx^i}{dt} = f^i\left(t, x^j\right) \qquad (i, j, \ldots = 1, 2, \ldots, n). \tag{1}$$

С этой системой связана три-ткань $W(1, n, 1)$, заданная на многообразии переменных $x^i$, $t$, состоящая из семейств $\lambda_\alpha$: $\lambda_1 : x^i = const$, $\lambda_2 : t = const$, $\lambda_3 :$

$F^i(t, x^j) = c^i = const$, причем последнее семейство состоит из интегральных кривых системы уравнений (1). Отметим, что три-ткани рассматриваются с точностью до локальных диффеоморфизмов на базах слоений, образующих ткань: $t = t(\tilde{t})$, $x^i = x^i(\tilde{x}^j)$ и $c^i = c^i(\tilde{c}^j)$, где $c^i$ — константы интегрирования.

Цель работы — интерпретировать свойства три-ткани в терминах дифференциальных уравнений и наоборот.

Изучение ведется методом Картана—Лаптева, то есть с три-тканью $W(1, n, 1)$ связывается семейство адаптированных реперов (корепероов). В этом репере первая серия структурных уравнений ткани имеет вид:

$$\begin{aligned} d\omega^u &= \omega^v \wedge \omega_v^u + \mu^u \omega^n \wedge \omega^{n+1}, \\ d\omega^n &= \omega^u \wedge \omega_u^n + \omega^n \wedge \omega_n^n, \\ d\omega^{n+1} &= \omega^{n+1} \wedge \omega_n^n, \end{aligned} \tag{2}$$

где $u, v, \ldots = 1, 2, \ldots, n-1$. Формы и компоненты относительного тензора $\{\mu^u\}$, входящие в эти уравнения, выражаются через частные производные от функций, определяющих систему (1).

**Теорема 1.** *Система обыкновенных дифференциальных уравнений автономна в том и только том случае, если* $\mu^u$ *и* $\omega_n^n$ *равны нулю.*

С геометрической точки зрения тензор $\{\mu^u\}$ является тензором неголономности некоторой неголономной три-ткани, связанной с рассматриваемой системой (с три-тканью $W(1, n, 1)$).

**Теорема 2.** *Пусть* $S$ *— почти автономная система ОДУ, для которой относительный инвариант* $t_n$ *обращается в нуль, и* $\tilde{W}(c^u)$ *— соответствующая ей двумерная регулярная три-ткань. Следующие условия эквивалентны:*

(а) *тензор* $\{t_u\}$ *обращается в нуль*;

(б) $d\omega_n^n = 0$;

(в) *уравнения тканей* $\tilde{W}(c^u)$ *приводятся к каноническому виду одновременно на всем многообразии* $M$;

(г) *существует допустимое преобразование, при котором система* $S$ *приводится к автономному виду.*

Здесь почти автономной мы называем систему, для которой выполняется условие $\mu^u = 0$, а $t_u$ и $t_n$ — величины, входящие во вторую серию структурных уравнений три-ткани $W(1, n, 1)$.

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# Авторезонанс в системах со слабой диссипацией

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Авторезонансом принято называть явление, которое случается в нелинейных осцилляторах под действием малых возмущений с переменной частотой накачки. Суть его состоит в значительном изменении амплитуды колебаний (либо энергии) системы благодаря резонансу, который автоматически поддерживается в течении длительного времени. В нелинейной физике известно много приложений этого явления. Нахождение условий существования таких состояний составляет основную задачу теории авторезонанса.

В докладе анализируются системы со слабой диссипацией на примере уравнений Ландау—Лифшица и уравнений Блоха. Эти уравнения принято использовать в качестве подходящих математических моделей в задачах магнитодинамики. Наличие диссипации в таких системах делает невозможным появление авторезонанса при постоянной амплитуде возмущения. Показано, что диссипативные потери можно компенсировать медленным ростом амплитуды возмущения. При этом рост амплитуды в решении обязан резонансу и определяется деформацией частоты накачки.

Получаемые результаты основаны на исследовании модельных неавтономных нелинейных уравнений главного резонанса. Для них построены решения с неограниченно растущей амплитудой, для которых доказана устойчивость по Ляпунову. Такие решения описывают начальный этап захвата в авторезонанс.

# Некоторые коэффициентные обратные задачи для параболических уравнений

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В докладе рассматриваются вопросы существования и единственности решений обратных задач определения одного из неизвестных коэффициентов в параболическом уравнении

$$\rho(t,x)u_t - a(t,x)u_{xx} + b(t,x)u_x + d(t,x)u = f(t,x), \quad (t,x) \in Q \equiv [0,T] \times [0,l]. \quad (1)$$

Предполагается, что заданы краевые условия

$$u(0,x) = u_0(x), x \in [0,l], \quad u(t,0) = \beta_1(t), u(t,l) = \beta_2(t), t \in [0,T], \quad (2)$$

а также дополнительное условие интегрального наблюдения

$$\int_0^T u(t,x)\chi(t)dt = \varphi(x). \quad (3)$$

В задачах (1)–(3) неизвестными являются функция $u(t,x)$, а также один из коэффициентов $a(t,x) \equiv p(x)$, $b(t,x) \equiv p(x)$ или $d(t,x) \equiv p(x)$, зависящий только от $x$.

Исследование существования и единственности решений рассматриваемых задач в докладе проводится в классах Соболева при минимальных требованиях гладкости на известные входные данные этих задач. Как оказалось, определяющую роль в таких исследованиях могут сыграть априорные оценки норм решений прямой задачи (1)–(2) в пространствах Соболева с явно вычисленными константами.

Например, вопрос о существовании решения обратной задачи может быть сведен к вопросу о разрешимости некоторого операторного уравнения

$$p = A(p) \tag{4}$$

в определенном банаховом пространстве, и знание таких констант позволяет указать условия на входные данные обратных задач (1)–(3), при которых оператор $A$ в уравнении (4) обладает требуемыми свойствами, например, является компактным или сжимающим.

Все условия доказываемых теорем существования и единственности решений обратных задач (1)–(3) выписываются в виде легко проверяемых неравенств. Приводятся нетривиальные примеры конкретных обратных задач, для которых такие условия выполнены, а следовательно, для них справедливы доказанные теоремы существования и единственности решения.

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# Решение задач для дифференциальных уравнений гиперболического типа методом характеристик

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Вывод формулы Даламбера для задачи Коши для одномерного волнового уравнения — метод характеристик решения задачи. Методом характеристик находится общее решение и решение задачи Коши для линейного дифференциального уравнения с частными производными первого порядка [1, с. 306–343]. Этим методом найдены решения задачи Коши для многих других дифференциальных уравнений, в том числе и нелинейных. Это уравнения Гамильтона—Якоби, квазилинейные уравнения первого порядка и другие [2, 3].

Для гиперболического уравнения порядка $m$ с постоянными коэффициентами в случае двух независимых переменных $t$ и $x$

$$\prod_{k=1}^{m}\left(\frac{\partial}{\partial t} - a^{(k)}\frac{\partial}{\partial x} + b^{(k)}\right)u(t,x) = f(t,x) \tag{1}$$

в аналитическом виде дается решение задачи Коши. В аналитическом виде получено решение задачи Коши и для одномерного телеграфного уравнения

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial t} + cu = f(t,x). \tag{2}$$

В полуполосе $Q = (0, \infty) \times (0, l)$ переменных $t, x$ найдены решения смешанных задач для строго и нестрого биволновых уравнений.

В $Q$ рассмотрены смешанные задачи для гиперболического уравнения вида (1) второго порядка.

Для одномерного волнового уравнения рассмотрены многоточечные задачи, задачи управления граничными условиями.

Рассмотрены некоторые задачи не только в полуполосе $Q$, но и в областях с криволинейными боковыми границами.

Метод построения решений смешанных и других задач состоит в следующем. С помощью характеристик рассматриваемая область, в которой задается основное уравнение, разбивается на подобласти. В каждой из подобластей из общего решения находим решение задачи, соответствующее исходной задаче. Эти решения на границах раздела согласовываем друг с другом таким образом, чтобы результирующее решение было достаточно гладким и удовлетворяло всем условиям исходной рассматриваемой задачи и ее уравнению.

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## Групповая классификация, симметрийная редукция и точные решения нелинейного уравнения колмогоровского типа


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Сообщение посвящено симметрийному анализу нелинейного уравнения

$$u_t - u_{xx} - uu_y = f(u), \tag{1}$$

которое нашло широкие приложения в задачах финансовой математики, теории диффузионных процессов, теории стохастического контроля [1, 2].

Получены следующие результаты.

(1) Проведена групповая классификация уравнения (1), согласно которой в общем случае оно инвариантно относительно трехмерной алгебры Ли, базис которой составляют операторы $P_0 = \partial_t, P_1 = \partial_x, P_2 = \partial_y$. Расширение симметрии возможно в следующих случаях (ниже приведены значения функции $f(u)$ в уравнении (1)):

- $exp(u); u^m (m \neq 0,1,2)$ — алгебры инвариантности четырехмерные;
- $u; u^2+1$ — алгебры инвариантности пятимерные;
- $u^2; 0; 1$ — алгебры инвариантности шестимерные.

(2) Для каждого из полученных инвариантных уравнений проведена классификация одно- и двухмерных подалгебр алгебр симметрии, которым соответствует симметрийная редукция к дифференциальным уравнениям с меньшим количеством независимых переменных (в частности, к обыкновенным дифференциальным уравнениям).

(3) Результаты симметрийной редукции использованы для построения ряда инвариантных решений уравнений вида (1).

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# Об оценках решений системы уравнений Прандтля для микронеоднородной стратифицированной магнитной жидкости


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Изучается поведение сильно стратифицированной магнитной жидкости. Малый параметр $\varepsilon > 0$ определяет толщину слоев жидкости. Согласно теории Прандтля, жидкость можно считать вязкой только в окрестности обтекаемого тела, где систему уравнений Навье—Стокса можно заменить более простой системой

$$\nu \frac{\partial^2 u_\varepsilon}{\partial y^2} - u_\varepsilon \frac{\partial u_\varepsilon}{\partial x} - v_\varepsilon \frac{\partial u_\varepsilon}{\partial y} = -\theta_\varepsilon \, (U^\infty - u_\varepsilon) - U^\infty \frac{dU^\infty}{dx}, \quad \frac{\partial u_\varepsilon}{\partial x} + \frac{\partial v_\varepsilon}{\partial y} = 0 \qquad (1)$$

в области $D = \{0 < x < X_0, 0 < y < \infty\}$ с граничными условиями

$$u_\varepsilon(0,y) = U(y), \quad u_\varepsilon(x,0) = 0, \; v_\varepsilon(x,0) = V_\varepsilon(x), \quad u_\varepsilon(x,y) \to U^\infty(x) \;\text{ при }\; y \to \infty. \qquad (2)$$

Здесь $\theta_\varepsilon(x,y) = \dfrac{\sigma_\varepsilon(x,y)B^2(x)}{\rho} > 0$, $\sigma_\varepsilon$ — магнитная проводимость жидкости, $B$ — ортогональная к поверхности обтекаемой пластины компонента вектора магнитной индукции, $\rho = 1$ — плотность жидкости, $(u_\varepsilon(x,y),\, v_\varepsilon(x,y))$ — поле скоростей потока жидкости (параллельная и ортогональная пластине компоненты, соответственно), $(U(y),0)$ — начальная скорость потока, $(0, V_\varepsilon(x))$ — скорость на нижней границе рассматриваемой области, $(U^\infty(x),0)$ — скорость на верхней границе.

Задача

$$\nu \frac{\partial^2 u}{\partial y^2} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = -\theta \, (U^\infty - u) - U^\infty \frac{dU^\infty}{dx}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

в области $D$ с граничными условиями

$$u(0,y) = U(y), \quad u(x,0) = 0, \; v(x,0) = V(x), \quad u(x,y) \to U^\infty(x) \text{ при } y \to \infty, \tag{4}$$

где

$$V_\varepsilon(x) \to V(x), \qquad \theta_\varepsilon(x,y) \to \theta(x,y) \quad \text{при} \quad \varepsilon \to 0,$$

является усредненной задачей для задачи (1) и имеет место следующее утверждение.

**Теорема.** *Пусть*

$$\max_{(x,y)\in D} |\theta_\varepsilon(x,y) - \theta(x,y)| \leqslant C_0\varepsilon, \qquad \max_{x\in[0,X_0]} \left| \int\limits_0^x (V_\varepsilon(x) - V(x))\, dx \right| \leqslant C_1\varepsilon.$$

*Тогда для любого* $N > 0$ *существует такое* $C > 0$*, что выполняется оценка*

$$\max_{(x,y)\in D_N} \left|\sqrt{u_\varepsilon} - \sqrt{u}\right| \leqslant C\varepsilon^{\frac{1}{2}},$$

*где* $D_N = D \cap \{y < N\}$*, где* $u_\varepsilon$ *удовлетворяет задаче* (1), (2)*, а* $u$ *удовлетворяет задаче* (3), (4).

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# Новые варианты принципа компенсированной компактности и их приложения

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В работах [1–6] были установлены новые варианты принципа компенсированной компактности, которые нашли применение в теории разрешимости различных эллиптических и параболических уравнений с нестандартными условиями роста, а также систем Навье—Стокса для несжимаемой неньютоновой жидкости, стационарной и нестационарной. Для указанных уравнений решение строится как предел приближений (галеркинских приближений или решений регуляризованной задачи). Для обоснования процедуры предельного перехода необходимо установить сходимость потоков к потоку. Здесь мы имеем дело с последовательностью потоков, которая представляет собой нелинейную функцию от слабо сходящейся последовательности приближенных градиентов. Для идентификации предела последовательности потоков недостаточно классических соображений монотонности. На этом этапе подходящие леммы о компенсированной компактности играют ключевую роль.

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# Стратегия гарантирующего управления спуском КА и вводом в действие исследовательских зондов в условиях неопределенности параметров атмосфер исследуемых планет

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В настоящее время при формировании проектного облика исследовательских КА наряду с задачами по изучению поверхности планет Солнечной системы особое внимание уделяется детальному изучению свойств атмосферы исследуемой планеты с помощью специальных зондов.

Например, в программе длительного изучения планеты Венера для контактных исследований атмосферы предполагается использовать атмосферный зонд — ветролет, принцип действия которого основан на использовании естественных условий атмосферы на планете: наличия постоянного ветра и существование устойчивого градиента ветра по высоте. Две аэродинамические поверхности, соединенные фалом разносятся на разные высоты, а изменение длины фала позволяет менять высоту дрейфа научной аппаратуры в атмосфере. Ввод в действие ветролета предполагается параллельно со спуском основного КА на поверхность планеты. При этом на условия ввода накладываются определенные ограничения, невыполнение которых из-за неучета относительно широких пределов неопределенности параметров атмосферы ведет к разрушению зонда, и, следовательно, к неудаче эксперимента.

В связи с этим предлагается применение гарантирующей стратегии управления спуском, в основе которой лежит «игровой» подход к формированию неопределенных воздействий на спускаемый аппарат в процессе его аэродинамического торможения.

Суть данного подхода заключается в предположении, что значение моделируемого фактора заключено в некоторых пределах так называемой области неопределенности, которая задается своими границами. В этом случае при решении каких-либо оптимизационных или предельных задач из всех возможных реализаций неопределенного фактора рассматривается та реализация (из области неопределенности), которая наихудшим образом скажется на исследуемых показателях (например, на критерии оптимальности).

Выполнение всех накладываемых ограничений при наихудших реализациях неопределенных воздействий будет означать гарантированное выполнение данных ограничений при любых возможных реализациях этих воздействий.

Для подтверждения гарантии искомого результата предлагается гипотетическое статистическое моделирование спуска, в основу которого положен комбинированный вариант учета неопределенности. В этом случае область неопределенности «заполняется» гипотетическими вероятностными характеристиками, полученными для подобных физических условий. Например, когда области неопределенности, характерные для параметров атмосферы планеты, «заполняются» вероятностными характеристиками поведения соответствующих земных параметров атмосферы.

# К проблеме усечения цепочки уравнений

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Во многих физических задачах возникает проблема *усечения* (обрыва) цепочки уравнений, моделирующей процесс. Вопрос в том, как определять корректность такого усечения. Это одна из стандартных задач для систем моментных аппроксимаций кинетических уравнений с бесконечной цепочкой уравнений. Для линеаризации в окрестности состояния равновесия 26-моментной системы Грэда, вычисленной для молекул Максвелла, будет доказано существование корректного по Чепману усечения смешанной задачи. Более того, будет показано, что переменными усечения являются гидродинамические переменные (плотность, скорость, температура) и тепловой поток, что дает возможность использовать как граничные условия для усечения физически оправданные краевые условия. Последнее позволило смоделировать эффекты плоского течения Куэтта.

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# Метод Лапласа в решении нелинейных уравнений Урысона

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Задачи дистанционного зондирования поверхности приводят к восстановлению решений систем нелинейных интегродифференциальных уравнений типа Урысона первого рода с дельтаобразными ядрами. Наибольший вклад в правую часть вносят критические точки, поэтому для приближенного решения этих уравнений применимы асимптотические методы. В первом приближении интегродифференциальный оператор (регуляризованное уравнение или регуляризирующий функционал) заменяется асимптотическим представлением, уточняется решение итерационно. В ряде прикладных задач достаточно ограничиться поиском только характерных точек решения (например, экстремальных). В этом случае, при наличии априорной информации, можно получить эффективные алгоритмы — например, в задаче восстановления характерных точек решения для системы интегродифференциальных уравнений первого рода вида

$$\int_a^b f_k(s, z(s), z'(s)) \exp\left(-\lambda(t - \frac{2}{c}R_k(z(s)))\right)^2 ds = u_k(t), \quad k = 1, 2,$$

$$R_k(z) = \left((M - z(s))^2 + (s - a_k)^2\right)^{\frac{1}{2}},$$

где $f_k > 0$ — достаточное число раз непрерывно дифференцируемая функция, $\lambda >> 1$ — параметр, $H >> 1$, $c$, $M$, $a_1$, $a_2$ — фиксированные числа, $R_k(z) \neq 0$. Предполагая существование решения системы $z(s)$ при заданных $u_k(t)$, $k = 1, 2$, находим решение из системы уравнений, получаемой заменой интегралов по методу Лапласа, и предположения о близости стационарных точек $z(s)$ и $R_k(z)$.

В качестве примера будем рассматривать нелинейное интегральное уравнение вида

$$\int_a^b f(s) n(t - z(s)) ds = u(t), \quad c \leqslant t \leqslant d, \tag{1}$$

где $f(s)$ — известная функция, $u(t)$ — заданная функция, $z(s)$ — искомая функция с ядром следующего вида:

$$n(t) = \sqrt{\frac{\beta}{\pi}} e^{-\beta t^2}. \tag{2}$$

Ядро является дельтаобразным: $n(t) \to \delta(t)$ при $\beta \to \infty$; это позволяет более эффективно решать задачу нахождения характеристических точек.

Применим к анализу уравнения метод Лапласа и найдем критические точки у неизвестной функции $z(s)$ по критическим точкам $u(t)$.

Обозначим $(t - z(s))^2$ через $S(t, s)$.

Различный вклад в (1), (2) вносят следующие точки:

(1) точки, в которых $S'(t, s) = 0$;

(2) концы промежутков интегрирования;

(3) точки, в которых $S(t, s) = 0$.

Наибольший вклад в правую часть уравнения (2) вносится, когда сама функция $S(t, s)$, и ее производные равны нулю.

$S(t, s) = (t - z(s))^2 = 0$, если $t - z(s) = 0$.

В зависимости от функции $z(s)$ выражение $t - z(s) = 0$ может не иметь решений, иметь одно или несколько решений.

Далее, $S'(t, s) = 2(t - z(s))z'(s) = 0$, если $t - z(s) = 0$ или $z'(s) = 0$.

В зависимости от точек $t$ возможны различные случаи использования асимптотических формул для решения исходного уравнения.

Заметим, что $S''(t, s) = -2z'^2(s) + 2(t - z(s))z''(s) = 0$, если $z'(s) = 0$; $\quad t - z(s) = 0$ или $z'(s) = 0$; $\quad z''(s) = 0$.

Если $t$ таково, что $S'(t, s) = 0$, то $S''(t, s) = 0$.

Продолжая по аналогии, получаем: $S^{(n)}(t, s) = -2nz'(s)z^{(n-1)}(s) + 2(t - z(s))z^{(n)}(s) = 0$, если $z'(s) = 0$; $\quad t - z(s) = 0$ или $z'(s) = 0$; $\quad z^{(n)}(s) = 0$, или $z^{(n-1)}(s) = 0$; $\quad z^{(n)}(s) = 0$, или $z^{(n-1)}(s) = 0$; $\quad t - z(s) = 0$.

В этом случае $S'(t, s) = S''(t, s) = \ldots = S^{(n)}(t, s) = 0$.

Для интегралов такого вида известно ( [1, с. 28]), что функция

$$\int_a^b f(s)e^{\lambda S(t,s)}ds = u(\lambda, t) \tag{3}$$

при $\lambda \to \infty$ имеет следующую асимптотику:

$$u(\lambda, t) = \sqrt{\frac{2\pi}{-\lambda S''(t, s_0)}}e^{\lambda S(s,t)}f(s_0, t) + o(\lambda^{-1}),$$

где $s_0$ — точка максимума $S(s, t)$.

Подынтегральная функция имеет при больших $\lambda$ резкий максимум (т. е. интеграл по отрезку $[a, b]$ можно приближенно заменить интегралом по малой окрестности точки максимума), и в окрестности точки максимума подынтегральную функцию можно заменить более простой.

Если $S(t, s) = (t - z(s))^2$, то $S''(t, s_0) = z'(s_0)^2$, причем $s_0$ такова, что $z(s_0) = t$ (предполагаем, что $z'(s_0) \neq 0$). Тогда

$$u(\lambda, t) \approx \frac{\sqrt{2\pi}}{\sqrt{\lambda}|z'(s_0|}f(s_0) + o(\lambda^{-1}).$$

Для $t : \min\limits_{a \leqslant s \leqslant b}(t - z(s))^2 \neq 0$ имеем соотношение $u(t) = o(\lambda^{-\infty})$.

Предположим, что для фиксированного $t$ существуют корни $s_1, s_2, \ldots, s_p$ уравнения $z(s) = t$ и все они являются простыми, причем ни одна из этих точек не является граничной точкой отрезка $[a,b]$. Тогда имеет место асимптотическое представление

$$u(t) = \lambda^{-\frac{1}{2}}\sqrt{\pi}\left(\sum_{k=1}^{p}\frac{f(s_k)}{z'(s_k)} + o(\lambda^{-1})\right).$$

Если для выбранного $t$ существует единственный кратный корень $\widehat{s}$ уравнения $z(s) = t$ кратности 1, такой что $z'(\widehat{s}) = 0, \quad z''(\widehat{s}) \neq 0$, то

$$u(t) = \lambda^{-\frac{1}{4}}\left(\frac{f(\widehat{s})}{\sqrt{|z''(\widehat{s})|}} \cdot \frac{1}{2}\Gamma(\frac{1}{4}) + o(\lambda^{-\frac{1}{2}})\right),$$

где $\Gamma(x)$ — гамма-функция.

При условии, что $f(s)$ — положительная непрерывная медленно меняющаяся функция, получаем, что функция $u(t)$ принимает максимальные значения в точках, лежащих в окрестности порядка $\lambda^{-\frac{1}{4}}$ точек $\widehat{t}$, для которых существует стационарная точка $\widehat{s}$ функции $z(s)$, такая что $z(\widehat{s}) = \widehat{t}$. Таким образом, с погрешностью $\varepsilon^{-\frac{1}{4}}$ функция $u(t)$ принимает максимальные значения в точках $\widehat{t}$, для которых существует стационарная точка $\widehat{s}$ функции $z(s)$ такая, что $z(\widehat{s}) = \widehat{t}$.

Аналогично, пусть $[a,b]$ — конечный отрезок и выполнены следующие условия:

(1) $\max\limits_{s\in[a,b]} S(t,s)$ достигается только в точке $s = a$;

(2) $f(s), S(t,s) \in C([a,b])$;

(3) $f(s), S(t,s) \in C^\infty$ при $s$, близких к $a$ и $S'(t,a) \neq 0$, т. е. $z(a) \neq t, z'(a) \neq 0$, при $\beta \to \infty, \quad \beta \in S_\varepsilon$ — $\varepsilon$-окрестность

$$u(t) \sim e^{\beta(t-z(a))}\sum_{k=0}^{\infty} c_k\beta^{-k-1}.$$

Коэффициенты $c_k$ имеют вид:

$$c_k = -M^k(\frac{f(s)}{S'(t,s)})|_{s=a}, \quad M = -\frac{1}{2z'(s)(t-z(s))}.$$

Это разложение можно дифференцировать по $\beta$ любое число раз.

Подобные асимптотические формулы имеют место и в двумерном случае. Эти формулы позволяют получать характерные точки решения и в других случаях.

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# Вычисление дифференциальных инвариантов обыкновенного дифференциального уравнения

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ОДУ $y' = f(x,y)$ определяет на плоскости адекватный геометрический объект — три-ткань, образованную декартовой сетью и интегральными кривыми этого уравнения. Мы рассматриваем три-ткани с точностью до замены параметров на базах слоений ткани. Каждый класс тканей — а, следовательно, и соответствующий класс ОДУ — характеризуется системой дифференциальных инвариантов. Они получаются следующим образом. С три-тканью каноническим образом ассоциируется некоторая аффинная связность, называемая связностью Черна. Кривизна этой связности $b$ и ее ковариантные производные разных порядков $b_1, b_2, b_{11}, b_{12}, b_{21}, b_{22}, \ldots$ являются относительными инвариантами. Из них строятся абсолютные инварианты. Кривизна $b$ выражается через частные производные до второго порядка включительно от функции $f$, ее ковариантные производные — через производные следующих порядков. Например, условие $b = 0$ выделяет класс ОДУ с разделяющимися переменными. Линейное уравнение $y' + yf(x) = g(x)$ характеризуется следующими соотношения на относительные инварианты: $bb_{22} - (b_2)^2 = 0, \quad bb_{21} - b_1 b_2 - b^3 = 0$; уравнение Риккати $y' = f(x)y^2 + g(x)y + h(x)$ — соотношением $b_{222}b - b_{22}b_2 = 0$.

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# Прямые и обратные спектральные задачи для оператора Штурма—Лиувилля в шкалах пространств Соболева

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Классическим наборам спектральных данных, которые обычно рассматриваются при решении обратных задач, мы ставим в соответствие специальные шкалы гильбертовых пространств, в которые помещаются эти спектральные данные. Это дает возможность корректно определить нелинейные отображения, ставящие в соответствие потенциалам из пространств Соболева наборы спектральных данных из построенных пространств. При такой постановке решение обратных задач сводится к точному описанию образа отображений. Однако язык теории отображений позволяет добиться новых результатов, в частности, получить равномерные априорные оценки для прямых и обратных отображений, которые ранее не были известны для классических задач. Полученные результаты позволяют решить проблему

В. А. Марченко приближенного восстановления потенциала по конечным наборам спектральных данных.

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